Lecture 2
The Wigner-Ville Distribution

Time-frequency analysis, adaptive filtering and source separation

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Time-Frequency representations

• The Short-Time Fourier Transform (STFT) takes a linear approach for a time-frequency representation

• It decomposes the signal on elementary components, called atoms

\[ h_{t,\omega}(\tau) = h(\tau - t)e^{-j\omega\tau} \]

• Each atom is obtained from the window \( h(t) \) by a translation in time and a translation in frequency (modulation)
Time-Frequency representations
Time-Frequency representations

• Alternatively, in the frequency domain

\[ H_{t,\omega}(\theta) = H(\omega - \theta)e^{-j\theta t} \]

• The STFT can be considered as the result of passing the signal through a bank of band-pass filters whose frequency response is \( H(\omega - \theta) \) and is deduced from a mother filter \( H(\omega) \) by a translation of \( \theta \)

• Each filter in the bank has a constant bandwidth
Time-Frequency representations

• If we consider the square modulus of the STFT, we get the spectrogram, which is the spectral energy density of the locally windowed signal $s_t(\tau) = s(\tau)h(\tau - t)$

• The spectrogram is a quadratic or bilinear representation

• If the energy of the windows is selected to be one, the energy of the spectrogram is equal to the energy of the signal

• Thus, it can be interpreted as a measure of the energy of the signal contained in the time-frequency domain centered on the point $(t, \omega)$
Time-Frequency representations

- Linear
  - STFT
  - Wavelet

- Bilinear or Quadratic
  - Cohen’s class
    - Spectrogram
    - Wigner-Ville
    - Choi-Williams
    - ...
  - Affine distributions
The energy distributions

- The purpose of the energy distributions is to distribute the energy of the signal over time and frequency.

- The energy of a signal $s(t)$ can be deduced from the squared modulus of either the signal or its Fourier transform:

$$E_s = \int |s(t)|^2 \, dt = \int |S(\omega)|^2 \, d\omega$$

- $|s(t)|^2$ and $|S(\omega)|^2$ can be interpreted as energy densities in time and frequency, respectively.
The energy distributions

• It is natural to look for a joint time and frequency energy density $\rho_S(t, \omega)$ such that

$$E_S = \iint \rho_S(t, \omega) dt d\omega$$

• As the energy is a quadratic function of the signal, the time-frequency energy distributions will be in general quadratic representations
The energy distributions

- Two properties that an energy density should satisfy are the time and frequency **marginal** conditions

\[
\int \rho_s(t, \omega) dt = |S(\omega)|^2
\]

\[
\int \rho_s(t, \omega) d\omega = |s(t)|^2
\]

- If the time-frequency energy density is integrated along one variable, the result is the energy density corresponding to the other variable
Cohen’s class

- There are many distributions that satisfy the properties mentioned before

- Therefore, it is possible to impose additional constraints to $\rho_s(t, \omega)$ that would result in desirable properties

- Among these properties, the covariance principles are of fundamental importance

- The Cohen’s class is the family of time-frequency energy distributions covariant by translations in time and frequency
Cohen’s class

- The spectrogram is an element of the Cohen’s class, since it is a quadratic, time- and frequency-covariant, and preserves energy

\[ x(t) = s(t - t_0) \Rightarrow P_x(t, \omega) = P_s(t - t_0, \omega) \]

\[ x(t) = s(t)e^{-j\omega_0 t} \Rightarrow P_x(t, \omega) = P_s(t, \omega - \omega_0) \]

\[ P(t, \omega) = |S(\omega)|^2 = |s(t)|^2 \]

- Taking the square modulus of an atomic decomposition is only a restrictive possibility to define a quadratic representation
The Wigner-Ville distribution

• The approach is based on the use of the autocorrelation function for calculating the power spectrum

• To construct the autocorrelation function, the signal is compared to itself for all possible relative shifts, or lags

\[ r_{ss}(\tau) = \int s(t)s(t + \tau)dt \]

where \( \tau \) is the shift of the signal with respect to itself
The Wigner-Ville Distribution

• In the standard autocorrelation function, time is integrated out of the result, and $r_{ss}$ is only a function of the time lag $\tau$.

• The Wigner-Ville (and all of Cohen’s class of distribution) uses a variation of the autocorrelation function where time remains in the result, called **instantaneous autocorrelation function**

$$R_{ss}(t, \tau) = s(t + \tau/2)s^*(t - \tau/2)$$

Where $\tau$ is the time lag and $^*$ represents the complex conjugate of the signal $s$. 
The Wigner-Ville Distribution

- Instantaneous autocorrelation of four cycle sine plots
The Wigner-Ville Distribution

• The Wigner-Ville Distribution (WVD) is defined as

\[
W_s(t, \omega) = \frac{1}{2\pi} \int s(t + \tau/2)s^*(t - \tau/2)e^{-j\omega\tau}d\tau
\]

or equivalently

\[
W_s(t, \omega) = \frac{1}{2\pi} \int S(\omega + \theta/2)S^*(t - \theta/2)e^{-j\theta t}d\theta
\]
The Wigner-Ville Distribution
The Wigner-Ville Distribution
The Wigner-Ville Distribution

- In an analogy to the STFT, the window is basically a shifted version of the same signal
- It is obtained by comparing the information of the signal with its own information at other times and frequencies
- It possesses several interesting properties, described as follows
Properties of the WVD

• **Energy conservation**: by integrating the WVD of $s$ all over the time-frequency plane, the energy of $s$ is obtained

$$E_S = \int \int W_s(t, \omega) dt \, d\omega$$

• **Real-valued**: the WVD is real-valued across time and frequency

$$W_s(t, \omega) \in \mathbb{R}, \quad \forall \ t, \omega$$
Properties of the WVD

- **Marginal properties**: the energy spectral density and the instantaneous power can be obtained as marginal distributions of $W_s$

\[
\int W_s(t, \omega)dt = |S(\omega)|^2
\]

\[
\int W_s(t, \omega)d\omega = |s(t)|^2
\]
Properties of the WVD

• **Translation covariance:** the WVD is time- and frequency-covariant

\[ x(t) = s(t - t_0) \Rightarrow W_x(t, \omega) = W_s(t - t_0, \omega) \]

\[ x(t) = s(t)e^{-j\omega_0 t} \Rightarrow W_x(t, \omega) = W_s(t, \omega - \omega_0) \]

• **Dilation covariance:** the WVD also preserves dilation

\[ x(t) = \sqrt{k}s(kt); \quad k > 0 \Rightarrow W_x(t, \omega) = W_s(kt, \omega/k) \]
Properties of the WVD

- **Compatibility with filterings**: it expresses the fact that if a signal \( x \) is the convolution of \( s \) and \( h \), the WVD of \( x \) is the time-convolution between the WVD of \( h \) and the WVD of \( s \)

\[
x(t) = \int s(\tau) h(t - \tau) d\tau \Rightarrow \\
W_x(t, \omega) = \int W_s(\tau, \omega) W_h(t - \tau, \omega) d\tau
\]
Properties of the WVD

- **Compatibility with modulations**: this is the dual property of the previous one: if \( x \) is the modulation of \( s \) by a function \( m \), the WVD of \( x \) is the frequency-convolution between the WVD of \( s \) and the WVD of \( m \)

\[
x(t) = s(t)m(t) \Rightarrow \quad W_x(t, \omega) = \int W_s(\tau, \theta)W_m(t, \omega - \theta)d\theta
\]
Properties of the WVD

• **Wide-sense support conservation**: if a signal has a compact support in time (respectively in frequency), then its WVD also has the same compact support in time (respectively in frequency). This is also called **weak finite support**

\[ s(t) = 0, |t| > T \Rightarrow W_s(t, \omega) = 0, |t| > T \]

\[ S(\omega) = 0, |\omega| > B \Rightarrow W_s(t, \omega) = 0, |\omega| > B \]

• However, the WVD does not have **strong finite support**
Properties of the WVD

- **Unitarity**: the unitarity property expresses the conservation of the scalar product from the time-domain to the time-frequency domain (apart from the squared)

\[
\left| \int s(t)x^*(t)dt \right|^2 = \iint W_s(t, \omega)W_x^*(t, \omega)dt\,d\omega
\]
Properties of the WVD

- **Instantaneous frequency and group delay**: The instantaneous frequency characterizes a local frequency behaviour as a function of time. In a dual way, the local time behaviour as a function of frequency is described by the group delay.

- In order to introduce these terms, the concept of **analytic signal** $s_a(t)$ must be defined first.
Properties of the WVD

• For any real valued signal $s(t)$, we associate a complex valued signal $s_a(t)$ defined as

$$s_a(t) = s(t) + jHT(s(t))$$

where $HT(s(t))$ is the **Hilbert transform** of $s(t)$

• $s_a(t)$ is called the analytic signal associated to $s(t)$
Properties of the WVD

• This definition has a simple interpretation in the frequency domain since $S_a$ is a single-sided Fourier transform where the negative frequency values have been removed, the strictly positive ones have been doubled, and the DC component is kept unchanged.

\[
S_a(\omega) = 0 \quad \text{if} \quad \omega < 0
\]

\[
S_a(\omega) = S(0) \quad \text{if} \quad \omega = 0
\]

\[
S_a(\omega) = 2S(\omega) \quad \text{if} \quad \omega > 0
\]
Properties of the WVD

• **Instantaneous frequency**: the instantaneous frequency of a signal $s$ can be recovered from the WVD as its first order moment (or center of gravity) in frequency

$$f_s(t) = \frac{\int \omega W_{sa}(t, \omega) d\omega}{\int W_{sa}(t, \omega) d\omega}$$
Properties of the WVD

• **Group delay**: the group delay of a signal $s$ can be recovered from the WVD as its first order moment (or center of gravity) in time

$$t_s(\omega) = \frac{\int tW_{sa}(t, \omega)dt}{\int W_{sa}(t, \omega)dt}$$
Interference in the WVD

• As the WVD is a bilinear function of the signal \( s \), the quadratic superposition principle applies

\[
W_{s+x}(t, \omega) = W_s(t, \omega) + W_x(t, \omega) + 2\Re\{W_{s,x}(t, \omega)\}
\]

where

\[
W_{s,x}(t, \omega) = \frac{1}{2\pi} \int s(t + \tau/2)x^*(t - \tau/2)e^{-j\omega\tau}d\tau
\]

is the cross-WVD of \( s \) and \( x \)
Interference in the WVD
Interference in the WVD
Interference in the WVD
Interference in the WVD

- These interference terms are troublesome since they may overlap with auto-terms (signal terms) and thus make it difficult to visually interpret the WVD image.

- It appears that these terms must be present or the good properties of the WVD (marginal properties, instantaneous frequency and group delay, localization, unitarity . . . ) cannot be satisfied.

- There is a trade-off between the quantity of interferences and the number of good properties.
Pseudo-WVD

• The definition of the WVD requires the knowledge of

\[ q_s(t, \omega) = s(t + \tau/2)s^*(t - \tau/2) \]

from \( t = -\infty \) to \( t = +\infty \), which can be a problem in practice

• Often a windowed version of \( q_s(t, \omega) \) is used, leading to the Pseudo-WVD (PWVD)

\[ PW_s(t, \omega) = \frac{1}{2\pi} \int s(t + \tau/2)s^*(t - \tau/2)h(t)e^{-j\omega\tau}d\tau \]
Pseudo-WVD

• This is equivalent to a frequency smoothing of the WVD, since

\[
P_{WS}(t, \omega) = \int H(\omega - \theta) W_S(t, \theta) \, d\theta
\]

where \( H(\omega) \) is the Fourier transform of \( h(t) \)

• Because of their oscillating nature, the interferences will be attenuated in the pseudo-WVD compared to the WVD
Pseudo-WVD
Pseudo-WVD

• However, the consequence of this improved readability is that many properties of the WVD are lost:
  • The marginal properties
  • The unitarity
  • The frequency-support conservation

• The frequency-widths of the auto-terms are increased by this operation
References and further reading


• Biosignal and Medical Image Processing, Second Edition by John L. Semmlow. CRC press; 2009. chapter 6 pp. 147-151

• The Time Frequency Toolbox tutorial (http://tftb.nongnu.org/tutorial.pdf)

• Material from Signals and System course, FI-UNER