# STOCHASTIC PROCESSES

# Exercises for Lecture 2

**Exercise 1.** Consider a motor neuron that discharges its first action potential at the time instant $t$ = 0. After this, it continues discharging with action potentials separated by intervals *Ti* which are random variables with a uniform probability density function between 100 ms and 150 ms. Compute the expected value of *T1* and the probability that *T1* is smaller than 120 ms.

**Solution**

$$\mu_X = E\{X\} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$E[Ti] = \int_{100}^{150} \frac{1}{50}\, x\, dx = 125\,\text{ms}$$

$$P(Ti < 120) = \int_{100}^{120} \frac{1}{50}\, dx = 0.4$$

**Exercise 2.** Consider the conditions of the previous exercise with a second motor neuron that discharges action potentials independently with respect to the first motor neuron but with the same probability density functions of the time intervals between discharges (identical but independent motor neuron). Compute the probability that the first discharge of the first motor neuron precedes in time the first discharge of the second motor neuron.

**Solution**

$$f(T_i^{st}, T_i^{nd}) = \left(\frac{1}{50}\right)^2 = \frac{1}{2500}$$

$$P(T_i^{st} < T_i^{nd}) = \frac{1}{2500} \iint_{T_i^{st} < T_i^{nd}} dT_i^{st} dT_i^{nd} = 0.5$$

The integral above should go from t1 = 100 to 150. And t2 = t1 to 150.

**Exercise 3.** Compute the first two moments (central and non central) of a random variable with uniform distribution between *a* and *b* (*b* > *a*).

**Solution**

$$m_k = E\{(x-\mu)^k\} = \int_a^b (x-\mu)^k \frac{1}{b-a} dx$$

**Exercise 4.** Given the function

$$f(x,y) = \begin{cases} 0.5(x+y) + kxy & 0 \le x \le 1; 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

with $k \in R$, find k such that $f(x,y)$ can be the joint pdf of X and Y. Are X and Y independent? Are they uncorrelated?

**Solution**

In order to be the joint pdf of X and Y, $f(x,y)$ must satisfy the two conditions:

$$\begin{cases} f(x,y) \ge 0 \\ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y) dx dy = 1 \end{cases}$$

that imply *k*=2.

X and Y are independent if $f(x,y) = f_X(x) f_Y(y)$

$$f_X(x) = \int_0^1 f(x,y)dy = \frac{3}{2}x + \frac{1}{4} \quad x \in [0,1]$$

$$f_Y(y) = \int_0^1 f(x,y)dx = \frac{3}{2}y + \frac{1}{4} \quad y \in [0,1]$$

$f(x,y) \neq f_X(x)f_Y(y)$, so X and Y are not independent.

If X and Y are uncorrelated, $E(XY) = E(X)E(Y)$

$$E(X) = E(Y) = \int_0^1 x\left(\frac{3}{2}x + \frac{1}{4}\right)dx = \frac{5}{8}$$

$$E(X)E(Y) = \frac{25}{64}$$

$$E(XY) = \int_0^1 \int_0^1 xyf(x,y)dxdy = \frac{7}{18}$$

$E(XY) \neq E(X)E(Y)$, so X and Y are not uncorrelated.

**Exercise 5**

The joint density function of two random variables *X* and *Y* is given by

$$f_{XY}(x,y) = \begin{cases} k(x+y), & 0 \le x \le 2 \quad \text{and} \quad 0 \le y \le 2 \\ 0 \quad , & \text{otherwise} \end{cases}$$

Find

(a) $k$

(b) The marginal density functions of *X* and *Y*

(c) $P(X < 1 | Y < 1)$

(d) $E[X]$, $E[Y]$, $E[XY]$

(e) Are *X* and *Y* independent?

**1.20** (*a*) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)dxdy = \int_0^2\int_0^2 k(x+y)dxdy = 1 \Rightarrow k = \frac{1}{8}$.

(*b*) The marginal density functions of *X* and *Y* are:

$$f_X(x) = \int_0^2 \frac{1}{8}(x+y)dy = \frac{x}{4} + \frac{1}{4} \quad \text{for } 0 \le x \le 2 \quad .$$

$$f_Y(y) = \int_0^2 \frac{1}{8}(x+y)dx = \frac{y}{4} + \frac{1}{4} \quad \text{for } 0 \le y \le 2 .$$

(*c*) $P(X < 1 \mid Y < 1) = \dfrac{\int_0^1\int_0^1 \frac{1}{8}(x+y)dxdy}{\int_0^1 (\frac{1}{4}y + \frac{1}{4})dy} = \dfrac{1/8}{3/8} = \dfrac{1}{3}$.

(*d*) $E[X] = \int_0^2 \frac{x}{4}(x+1)dx = \frac{7}{6} = E[Y]$.

$$E[XY] = \int_0^2\int_0^2 \frac{xy}{8}(x+y)dxdy = \frac{4}{3} .$$

(*e*) We observe from (*d*) that *X* and *Y* are correlated and thus, they are not independent.