# STOCHASTIC PROCESSES

# Exercises for Lecture 3

**Exercise 1.** The joint pdf of the random variables X and Y is given by :

$$f_{X,Y}(x,y) = \begin{cases} x+y & 0 \le x \le 1; 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

Determine the covariance of X and Y.

**Solution**

$$\sigma_{XY} = E\{(X-\mu_X)(Y-\mu_Y)\} = E\{XY\} - \mu_X\mu_Y$$

$$\mu_X = \mu_Y = \int_0^1 \int_0^1 x(x+y)dxdy = \frac{7}{12}$$

$$E\{XY\} = \int_0^1 \int_0^1 xy(x+y)dxdy = \frac{1}{3}$$

$$\sigma_{XY} = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = \frac{1}{144}$$

**Exercise 2.** Consider an EEG signal detected with additive noise. The noise has uniform probability density function of the amplitude (the probability density function does not change with time), with maximum amplitude ±A. The variance of the EEG signal does not change over time and is equal to 1. The EEG signal has zero mean. The EEG signal and the noise are independent. Find the variance of the detected signal (EEG plus noise).

**Solution**

The stochastic process is

$$Y = X + N$$

The noise has uniform distribution with

$$E\{N\} = \int_{-A}^{A} n\, f_N(n)\, dn = \frac{1}{2A}\int_{-A}^{A} n\, dn = 0$$

And

$$E\{N^2\} = \int_{-A}^{A} n^2 f_N(n) dn = \frac{1}{2A}\int_{-A}^{A} n^2\, dn = \frac{A^2}{3}$$

The variance of the process is:

$$E\left\{(Y-\mu_Y)^2\right\}=E\left\{Y^2\right\}=$$
$$E\left\{(X+N)^2\right\}=E\left\{X^2\right\}+E\left\{N^2\right\}+2E\{X\}E\{N\}=$$
$$1+\frac{A^2}{3}$$

**Exercise 3.** Consider a stochastic process $X$(t) with uniform probability density function of the amplitude (the probability density function does not change with time), with maximum amplitude ±A. The samples of $X$(t) are independent from each other. Compute mean, autocorrelation, autocovariance, and variance of the stochastic process $Y(t)=X(t)\sin(2\pi f_0 t+\phi)$ for the cases in which 1) $\phi$ is a deterministic value and 2) $\phi$ is a random variable with uniform distribution in the interval [-π,π]. $\phi$ is independent of X(t).

**Solution**

**1)**

The **mean** of the stochastic process is:

$$\mu_Y=E\{Y(t)\}=E\{X(t)\sin(2\pi f_0 t+\phi)\}=$$
$$E\{X(t)\}\sin(2\pi f_0 t+\phi)=0$$

The **autocorrelation** is

$$R_{YY}(t_1,t_2)=E\{Y(t_1)Y(t_2)\}=E\{X(t_1)X(t_2)\}\sin(2\pi f_0 t_1+\phi)\sin(2\pi f_0 t_2+\phi)=$$
$$=R_{XX}(t_1,t_2)\sin(2\pi f_0 t_1+\phi)\sin(2\pi f_0 t_2+\phi)$$

The **autocovariance** is given by

$$K_{YY}(t_1,t_2)=R_{YY}(t_1,t_2)-\mu_Y(t_1)\mu_Y(t_2)=R_{YY}(t_1,t_2)$$

The **variance** is

$$\sigma_Y^2=E\left\{(Y-\mu_Y)^2\right\}=R_{XX}(t,t)\sin^2(2\pi f_0 t+\phi)=\sigma_X^2\sin^2(2\pi f_0 t+\phi)=\frac{A^2}{3}\sin^2(2\pi f_0 t+\phi)$$

**2)**

With the assumption of independency

$$\mu_Y=E\{Y(t)\}=E\{X(t)\}E\{\sin(2\pi f_0 t+\phi)\}=0$$

The **autocorrelation** is

$$R_{YY}(t_1,t_2) = E\{Y(t_1)Y(t_2)\} = E\{X(t_1)X(t_2)\sin(2\pi f_0 t_1 + \phi)\sin(2\pi f_0 t_2 + \phi)\} =$$
$$= \frac{1}{2}R_{XX}(t_1,t_2)E\{\cos(2\pi f_0(t_1 - t_2)) - \cos(2\pi f_0(t_1 + t_2) + 2\phi)\} =$$
$$= \frac{1}{2}R_{XX}(t_1,t_2)E\{\cos(2\pi f_0(t_1 - t_2))\} - \frac{1}{2}R_{XX}(t_1,t_2)E\{\cos(2\pi f_0(t_1 + t_2) + 2\phi)\} =$$
$$= \frac{1}{2}R_{XX}(t_1,t_2)E\{\cos(2\pi f_0(t_1 - t_2))\}$$

The **autocovariance** is given by

$$K_{YY}(t_1,t_2) = R_{YY}(t_1,t_2) - \mu_Y(t_1)\mu_Y(t_2) = R_{YY}(t_1,t_2)$$

The **variance** is

$$\sigma_Y^2 = R_{YY}(t,t) = \frac{1}{2}\sigma_X^2$$

**Exercise 4.** Consider an EMG signal detected with additive noise. The EMG and the noise are independent, zero mean, WSS stochastic processes. The EMG signal has known autocorrelation function $R_X(\tau)$ and the noise has autocorrelation function $R_N(\tau)=\delta(t)$. Determine the autocorrelation function of the summation of the EMG signal and noise.

**Solution**

$$Y(t) = X(t) + N(t)$$

$$R_Y = E[Y(t)Y(t+\tau)] =$$
$$= E[(X(t) + N(t))(X(t+\tau) + N(t+\tau))] =$$
$$= E[X(t)X(t+\tau)] + E[X(t)N(t+\tau)] + E[X(t+\tau)N(t)] + E[N(t)N(t+\tau)] =$$
$$= R_X(\tau) + R_N(\tau) = R_X(\tau) + \delta(t)$$

(The EMG and the noise are independent, so $E[X(t)N(t+\tau)] = E[X(t)]E[N(t+\tau)] = 0$, $E[X(t+\tau)N(t)] = E[X(t+\tau)]E[N(t)] = 0$ )