STOCHASTIC PROCESSES

Solutions for Lecture 4

Exercise 1. Determine if the stochastic process $X(t) = A\sin(2\pi f_0 t + \phi)$, with ϕ random variable with uniform distribution in the interval [-π,π], is wide sense stationary.

Solution

A random process $X(t)$ is said to be wide sense stationary if its mean is independent of time and the autocorrelation function depends only on the time difference.

$$E[X(t)] = AE[\sin(2\pi f_0 t + \phi)] =$$
$$= A\int_{-\pi}^{+\pi} \sin(2\pi f_0 t + \phi)\frac{1}{2\pi}\,d\phi =$$
$$= -\frac{A}{2\pi}\cos(2\pi f_0 t + \phi)\Big|_{-\pi}^{+\pi} =$$
$$= \frac{A}{2\pi}\left(\cos(2\pi f_0 t - \pi) - \cos(2\pi f_0 t + \pi)\right) = 0$$

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] =$$
$$= A^2 E[\sin(2\pi f_0 t_1 + \phi)\sin(2\pi f_0 t_2 + \phi)] =$$
$$= \frac{A^2}{2} E[\cos(2\pi f_0 t_1 - 2\pi f_0 t_2) - \cos(2\pi f_0 t_1 + 2\pi f_0 t_2 + 2\phi)] =$$
$$= \frac{A^2}{2}\cos(2\pi f_0 (t_1 - t_2)) = R_X(t_1 - t_2)$$

The process is wide sense stationary.

Exercise 2. Consider an EEG signal modeled as a zero mean, WSS stochastic process with autocorrelation function equal to zero for $|\tau| > T$ and triangular (maximum value equal to 1) for $|\tau| \le T$. The signal is filtered by a filter with impulse response $h(t) = \delta(t) + \delta(t+T)$. Find the mean, the variance, and autocorrelation function of the output signal.

Solution

$$Y(t) = X(t) * (\delta(t) + \delta(t+T)) =$$
$$= X(t) + X(t+T)$$

$$E[Y(t)] = E[X(t)] + E[X(t+T)] = 0$$

$$R_{YY}(\tau) = E[(X(t) + X(t+T))(X(t+\tau) + X(t+T+\tau))] =$$
$$= E[X(t)X(t+\tau)] + E[X(t)X(t+T+\tau)] + E[X(t+T)X(t+\tau)] + E[X(t+T)X(t+T+\tau)] =$$
$$= 2R_X(\tau) + R_X(\tau+T) + R_X(\tau-T)$$

$$\sigma_Y^2 = E\left[Y(t)^2\right] - E[Y(t)]^2 = R_Y(0) - 0 = 2R_X(0) + R_X(T) + R_X(-T) = 2 + 0 + 0 = 2$$

Exercise 3. Consider an EMG signal with known autocorrelation function $R_X(\tau)$ and zero mean. The signal is filtered by a filter with impulse response $h(t) = (\delta(t) - \delta(t+T))/T$. Find the mean and autocorrelation function of the output signal.

Solution

$$Y(t) = X(t) * \frac{1}{T}(\delta(t) - \delta(t+T)) =$$
$$= \frac{1}{T}X(t) - \frac{1}{T}X(t+T)$$

$$E[Y(t)] = \frac{1}{T}E[X(t)] - \frac{1}{T}E[X(t+T)] = 0$$

$$R_{YY}(\tau) = \frac{1}{T^2}E[(X(t) - X(t+T))(X(t+\tau) - X(t+T+\tau))] =$$
$$= \frac{1}{T^2}(E[X(t)X(t+\tau)] - E[X(t)X(t+T+\tau)] - E[X(t+T)X(t+\tau)] + E[X(t+T)X(t+T+\tau)]) =$$
$$= \frac{1}{T^2}(2R_X(\tau) - R_X(\tau+T) - R_X(\tau-T))$$