AALBORG UNIVERSITY

Welcome to Stochastic Processes 1

Course Plan

Part 1: Probability concepts, random variables and random processes

Lecturer: Ernest Nlandu Kamavuako

Lecture 1: Introduction to probability and random variables
Lecture 2: Moments and multi dimensional random variables
Lecture 3: Introduction to random processes
Lecture 4: Stationarity

We expect the student to have a deep learning

Course Plan

Part 2: Random processes

Lecturer: Samuel Schmidt

Lecture 5: Some random processes (basic knowledge)
Lecture 6: Power spectral density
Lecture 7: Linear time-invariant systems (Filtering)
Lecture 8: Ergodicity

Lecture 9: Covariance matrix (if time allows)

Lecture 6 - 8: Deep learning

Learning outcomes

Outcomes:

- Have knowledge on stochastic processes in general
- Have knowledge about cross- and auto-correlation of stochastic processes.
- Have deep knowledge about power spectral analysis of stationary stochastic processes and ergodicity.
- Explain the defining properties of various stochastic processes

Exam: Written

Lecture 1: Introduction to probability and random variables

A deterministic signal can be derived by mathematical expressions.

A deterministic *model (or system)* will always produce the same output from a given starting condition or initial state.

In this course: stochastic (random) signals or processes

- Counterpart to a deterministic process
- Described in a probabilistic way
- Given initial condition, many realizations of the process

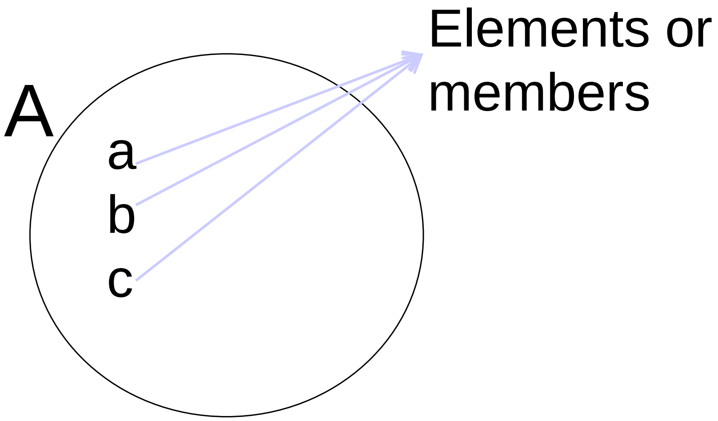
Teaching style: lectures (2 h) and exercises (2 h)

Concepts of probability

A prequisite for understanding the main content of the course.

Some definitions:
A *set* is a collecton of objects. $a \in A$

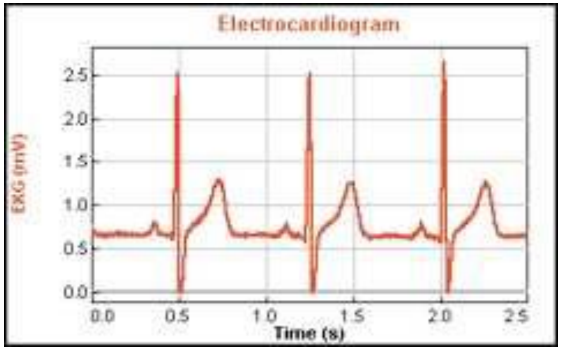
Given 2 sets *A* and *B*,
Intersection: $A \cap B$
Union: $A \cup B$



Probability theory

Probability provides mathematical models for random phenomena and experiments.

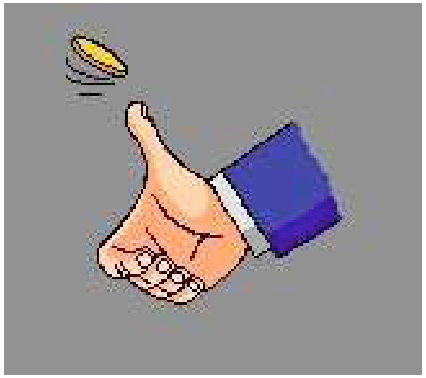
Some random phenomena



Its origin lies in observations associated with games of chance

Some definitions

Sample space S is a set of all possible distinct outcomes of interest in a particular experiment



Tossing a coin. S = {head, tail}
Toss a fair coin ***n*** times and heads come up $\boldsymbol{n_H}$ times.

The relative frequency of heads is $\boldsymbol{n_H/n}$ and will be very closed to **1/2**

An ***EVENT*** is a particular outcome or a combination of outcomes

Classical definition of probability

Define the probability of an event A as:

$$P(A) = \frac{N_A}{N}$$

where $\boldsymbol{N}$ is the number of possible outcomes of the random experiment and $\boldsymbol{N_A}$ is the number of outcomes favorable to the event A.

For example:

A 6-sided die has 6 outcomes.

3 of them are even,

Thus *P(even)* = 3/6

Problems with this classical definition

- Here the assumption is that all outcomes are equally likely (probable). Thus, the concept of probability is used to define probability itself! Cannot be used as basis for a mathematical theory.
- In many random experiments, the outcomes are not equally likely.
- The definition doesn't work when the number of possible outcomes is infinite.

Relative frequency definition of probability

The probability of en event *A* is defined as:

$$P(A) = \lim_{n\to\infty} \frac{n_A}{n}$$

where n_A is the number of times A occurs in n performances of the experiment

An experiment has been conducted and the probability is defined based on the observations.

Problems with Relative frequency definition of probability

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n}$$

- How do we assert that such limit exists?
- We often cannot perform the experiment multiple times (e.g. Limited time in the lab)

Axiomatic Definition of Probability

It provides rules for assigning probabilities to events in a mathematically consistent way

Elements of axiomatic definition:

- Set of all possible outcomes of the random experiment S (Sample space)
- Set of events, which are subsets of S

Axiomatic Definition of Probability

- A probability law (measure or function) that assigns probabilities to events such that:

  - $P(A) \geq 0$
  - $P(S) = 1$
  - If A and B are disjoint events (mutually exclusive), $i.e.\ A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$

That is if *A* happens, *B* cannot occur.

Advantages

- The theory provides mathematically precise ways for dealing with experiments with infinite number of possible outcomes, defining random variables, etc.
- The theory does not deal with what the values of the probabilities are or how they are obtained. Any assignment of probabilities that satisfies the axioms is legitimate.

Some Useful Properties

- $0 \leq P(A) \leq 1$
- $P(\emptyset) = 0$: probability of impossible event
- $P(\bar{A}) = 1 - P(A)$, $\bar{A}$ the complement of A
- If A and B are two events, then
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
- If the sample space consits of n mutually exclusive events such that $S = A_1 \cup A_2 \cup \cdots \cup A_n$, then
$$P(S) = P(A_1) + P(A_2) + \cdots + P(A_n) = 1$$

Joint and marginal probability

Joint probability:

is the likelihood of two events occurring together. Joint probability is the probability of event ***A*** occurring at the same time event ***B*** occurs. It is $P(A \cap B)$ or $P(AB)$.

Marginal probability:

is the probability of one event, ignoring any information about the other event. Thus $P(A)$ and $P(B)$ are marginal probabilities of events ***A*** and ***B***

Conditional probability

Let ***A*** and ***B*** be two events. The probability of event ***B*** given that event ***A*** has occured is called the *conditional probability*

$$\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

If the occurance of ***B*** has no effect on ***A***, we say ***A*** and ***B*** are *indenpendent* events. In this case $\boldsymbol{P(A|B) = P(A)}$

Combining both, we get $\boldsymbol{P(A \cap B) = P(A)P(B)}$, when ***A*** and ***B*** are indenpendent

Relationship involving Joint, marginal and conditional Probabilities

- $P(AB) = P(A|B)P(B) = P(B|A)P(A)$

- If $AB = \emptyset, then\ P(AUB|C) = P(A|C) + P(B|C)$

- If $B_1, B_2, \cdots, B_n$ are mutually exclusive and exhaustive events, then

$$P(A) = \sum_{j=1}^{n} P(A|B_j)P(B_j)$$

Example

An automatic diagnostic system classifies patients into:
Event *A*: the patient has the disease
Event *B*: the patient is healthy.

The probability that a patient belongs to group *A* is $P(A) = 0.05$ and that of group *B* is $P(B) = 0.95$.
The probability of the system to make an error is $P(E) = 0.01$.

Question 1: Find the probability of *A* detected with error: $P(A,E)$
Question 2: Let $P(E|A) = 0.02$ and $P(E|B) = 0.1$,
Find $P(A,E)$ and $P(B,E)$

Random variables

A *random variable* is a function, which maps **events or outcomes** (e.g., the possible results of rolling two dice: {1, 1}, {1, 2}, etc.) **to real numbers** (e.g., their sum).

A random variable can be thought of as a quantity whose value is not fixed, but which can take on different values.

A *probability distribution* is used to describe the probabilities of different values occurring.

Random variables

Notations:

Random variables with capital letters: ***X, Y, ..., Z***

Real value of the random variable by lowercase letters (x, y, …, z)

Types:

Continuous random variables: maps outcomes to values of an uncountable set. the probability of any specific value is zero

Discrete random variable: maps outcomes to values of a countable set. Each value has probability ≥ 0. $P(x_i) = P(X = x_i)$

Mixed random variables

Continuous random variables

Distribution function:

By definition

$$F_X(x) \triangleq P(X \leq x)$$

Properties:

1. $F_X(x)$ is either increasing or remains constant.

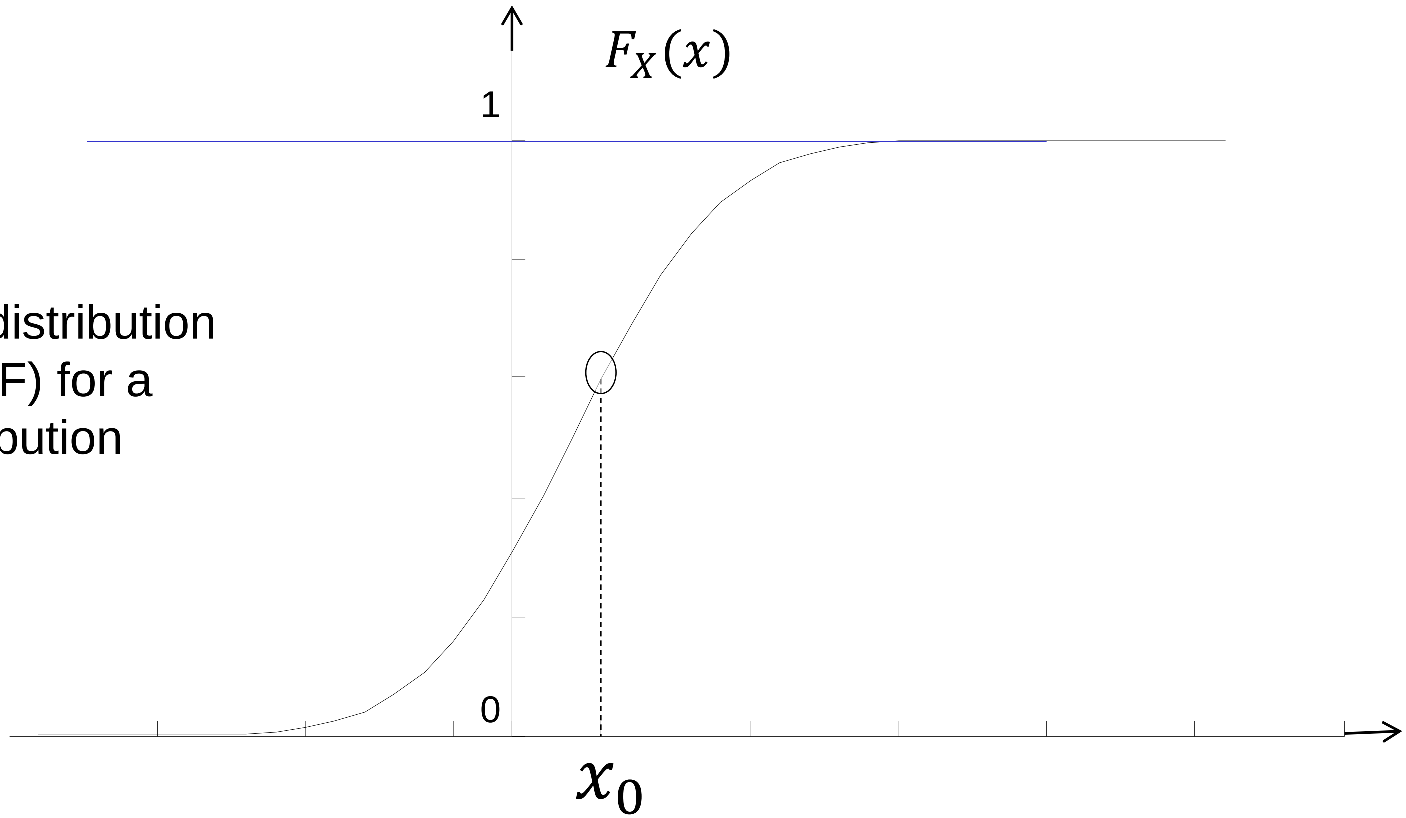
2. $\lim_{x \to -\infty} F_X(x) = 0$ 3. $\lim_{x \to +\infty} F_X(x) = 1$

4. $F_X(x_1) \leq F_X(x_2)\ if\ x_1 \leq x_2$

5. $P(a \leq X \leq b) = F_X(b) - F_X(a)$

Distribution functions

Cumulative distribution function (CDF) for a normal distribution



Probability density function (pdf)

Definition:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Properties:

1. $f_X(x) \geq 0$

2. $\int_a^b f_X(x)\,dx = F_X(x)|_a^b = F_X(b) - F_X(a) = P(a \leq X \leq b)$

3. $\int_{-\infty}^{+\infty} f_X(x)\,dx = F_X(x)|_{-\infty}^{+\infty} = 1$

Thus integration of $f_X(x)$ gives probability

Example

We have observed the firing of action potentials (AP) in different experiments and noticed that:

1. a motor neuron discharges its first AP at the time instant t = 0.
2. The second AP happens between 100ms and 200ms with T describing its position.
3. All have the same probability: uniform distribution

Question: Find the cdf and the pdf

Take home message

1. Some definitions in probability: sample space, event, set, etc...
2. 3 definitions of probability: Classical, relative frequency and axiomial
3. Joint, marginal and conditional probability
4. Random variables:
   a. Definition
   b. Cumulative distribution function (CDF)
   c. Probability dnesity function (PDF)
   d. Properties of CDF and PDF

Exercises

Course website:

http://person.hst.aau.dk/enk/ST6

If requested:
Username: sp1
Password: st6