AALBORG UNIVERSITY

Welcome to Stochastic Processes 1

Lecture 2:
Moments and multi dimensional random variables

Expectation operator

$$E\{g(X)\} \triangleq \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$$

Is defined only for random variables or a function of them.

Expected value: Is the weighted average of all possible values that this random variable can take on.

Let $g(X) = X$

$$E\{X\} \triangleq \int_{-\infty}^{+\infty} x f_X(x)\, dx$$

Example 1

Last time: A second action potential is firing between 100 and 200 ms. We found:

$$f_T(t) = \begin{cases} 0, & t < 100 \; and \; t > 200 \\ \frac{1}{100}, & 100 \le t \le 200 \end{cases}$$

Thus

$$E\{T\} = \int_{-\infty}^{+\infty} t f_T(t)\, dt = \int_{100}^{200} t \frac{1}{100} dt = 150\, ms$$

Definition of moments

A moment of order *k* of a random variable is defined as:

$$\mu_k = E\{X^k\} = \int_{-\infty}^{+\infty} x^k f_X(x)dx$$

Order 1: $\mu_1 = E\{X\} = \int_{-\infty}^{+\infty} x f_X(x)dx$ Mean of X

Order 2: $\mu_2 = E\{X^2\} = \int_{-\infty}^{+\infty} x^2 f_X(x)dx$

Central moments

Defined in a similar way as moment

$$m_k = E\{(X - E\{X\})^k\} = \int_{-\infty}^{+\infty} (x - \mu_1)^k f_X(x) dx$$

Variance of X

$$m_2 = E\{(X - \mu_1)^2\} = \int_{-\infty}^{+\infty} (x - \mu_1)^2 f_X(x) dx = \sigma^2$$

$$m_2 = E\{(X - \mu_1)^2\} = \mathrm{E}\{X^2\} - (E\{X\})^2$$

Two random variables

Let X and Y be two random variables, we define the *joint distribution function*

$$F_{XY}(x,y) \triangleq P(X \leq x, Y \leq y)$$

and the *joint probability density function* as

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \geq 0$$

Useful Properties

- $\lim_{x \to +\infty} F_{XY}(x, y) = F_Y(y)$
- $\lim_{y \to +\infty} F_{XY}(x, y) = F_x(x)$
- $\int_{-\infty}^{+\infty} f_{XY}(x, y)dx = f_Y(y)$
- $\int_{-\infty}^{+\infty} f_{XY}(x, y)dy = f_X(x)$
- $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{XY}(x, y) = 1$

The probability that X lies between x_1 and x_2 and Y lies between y_1 and y_2 is

$$P(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{Xy}(x, y)dxdy$$

Indenpendent and uncorrelated random variables

Let X and Y be two random variables, we say that

X and Y are *indenpendent* if

$$F_{XY}(x,y) = F_X(x) \cdot F_Y(y) \Rightarrow f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

X and Y are *uncorrelated* if

$$E\{XY\} = E\{X\} \cdot E\{Y\}$$

Indenpendence => uncorrelation

Example 2

Consider now that 2 cells (T_1 and T_2) are firing with the same probability density function as in example 1.

Calculate the probability that cell 1 discharges between 130 and 140 ms and cell 2 between 180 and 200 ms.

$$P(130 \le T_1 \le 140, 180 \le T_2 \le 200) = ?$$

Expected value

$$E\{g(X,Y)\} = \iint_{-\infty}^{+\infty} g(x,y) f_{XY}(x,y)\,dx\,dy$$

$$E\{XY\} = \iint_{-\infty}^{+\infty} xy f_{XY}(x,y)\,dx\,dy$$

The correlation

If X and Y are indenpendent we get

$$E\{XY\} = \iint_{-\infty}^{+\infty} xy f_X(x) f_Y(y)\,dx\,dy$$

$$E\{XY\} = \iint_{-\infty}^{+\infty} xy f_X(x) f_Y(y) dx dy$$

$$= \int_{-\infty}^{+\infty} x f_X(x) dx \cdot \int_{-\infty}^{+\infty} y f_Y(y) dy$$

$$= E\{X\} \cdot E\{Y\}$$

Uncorrelated

If $E\{XY\} = 0$ we say *X* and *Y* are *orthogonal*

$$E[X+Y] = E[X] + E[Y] \qquad m_{10} = E[X] = m_x \qquad m_{01} = E[Y] = m_y$$

$$\mu_{20} = E[(X - m_x)^2] = \sigma_x^2 \qquad \mu_{02} = E[(Y - m_y)^2] = \sigma_y^2$$

$$\rho_{xy} = \frac{E[(X - m_x)(Y - m_y)]}{\sigma_x \sigma_y}$$

Correlation coefficient

$$C_{xy} = E[(X - m_x)(Y - m_y)]$$

Covariance

CONTINUOUS DISTRIBUTION FUNCTIONS

The Uniform Distribution

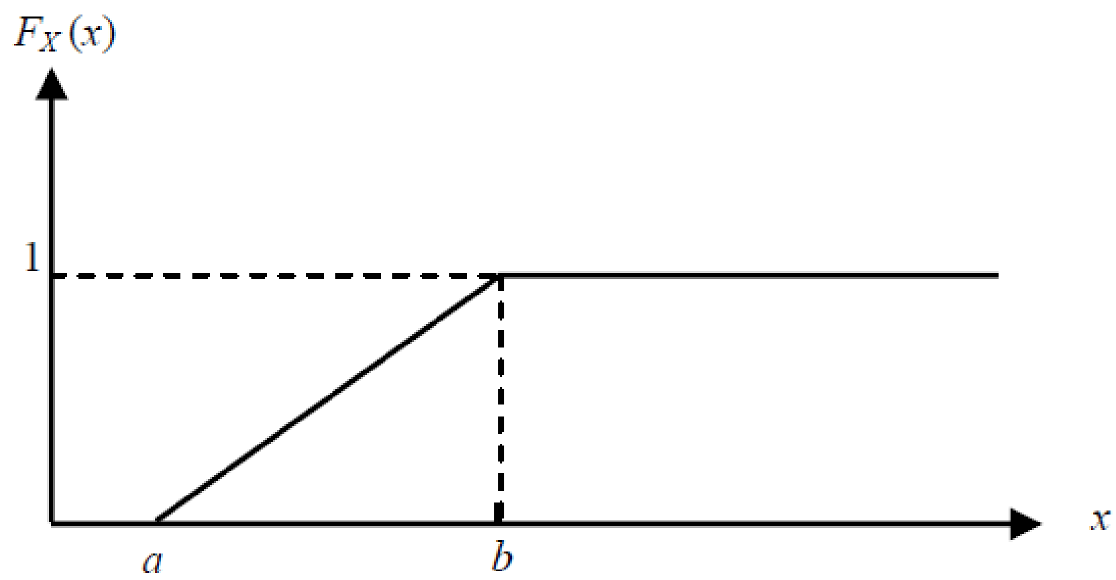
The probability density function is given as

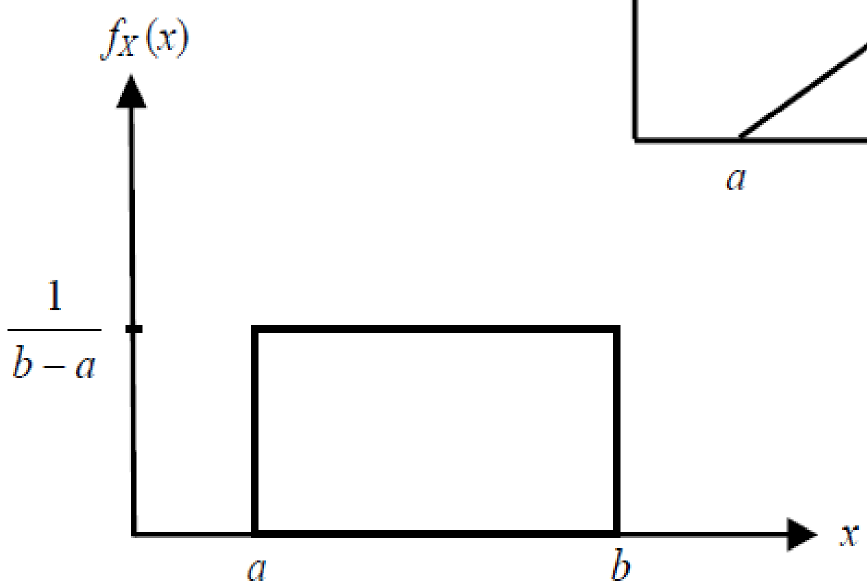
$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & otherwise \end{cases}$$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(u)\,du = \begin{cases} 0 \quad , & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ 1 \quad , & x \ge b \end{cases}$$

Uniform distribution

$$E[X] = \frac{1}{2}(a+b)$$

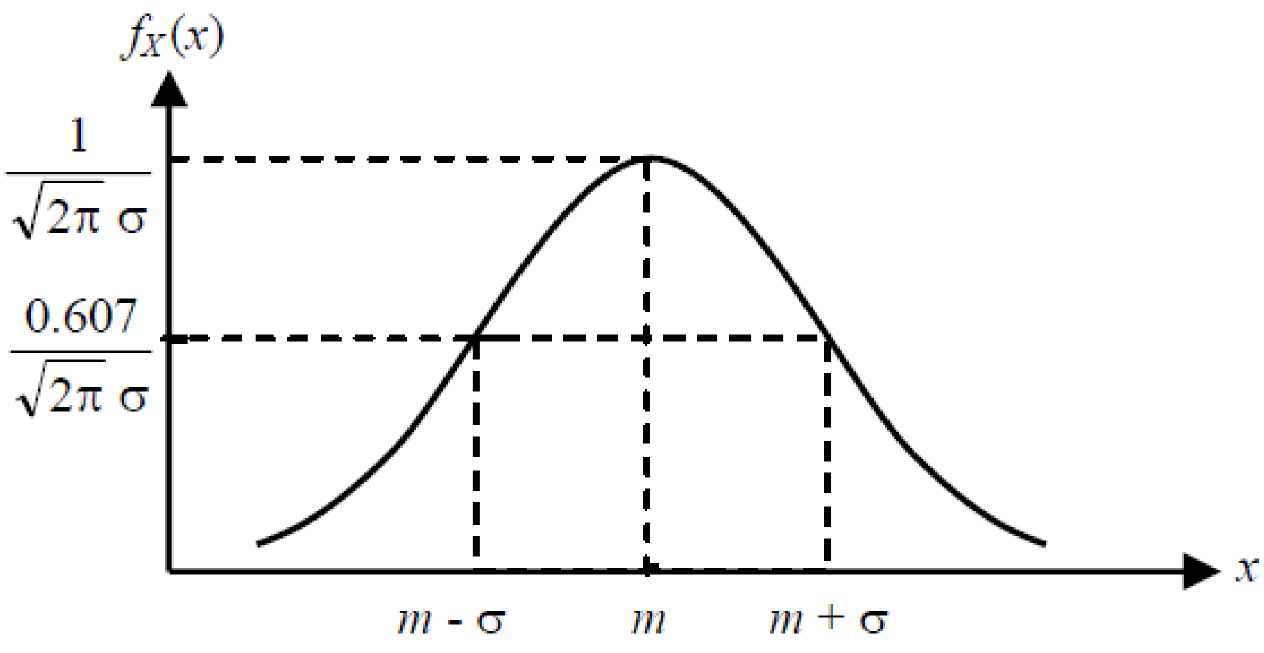




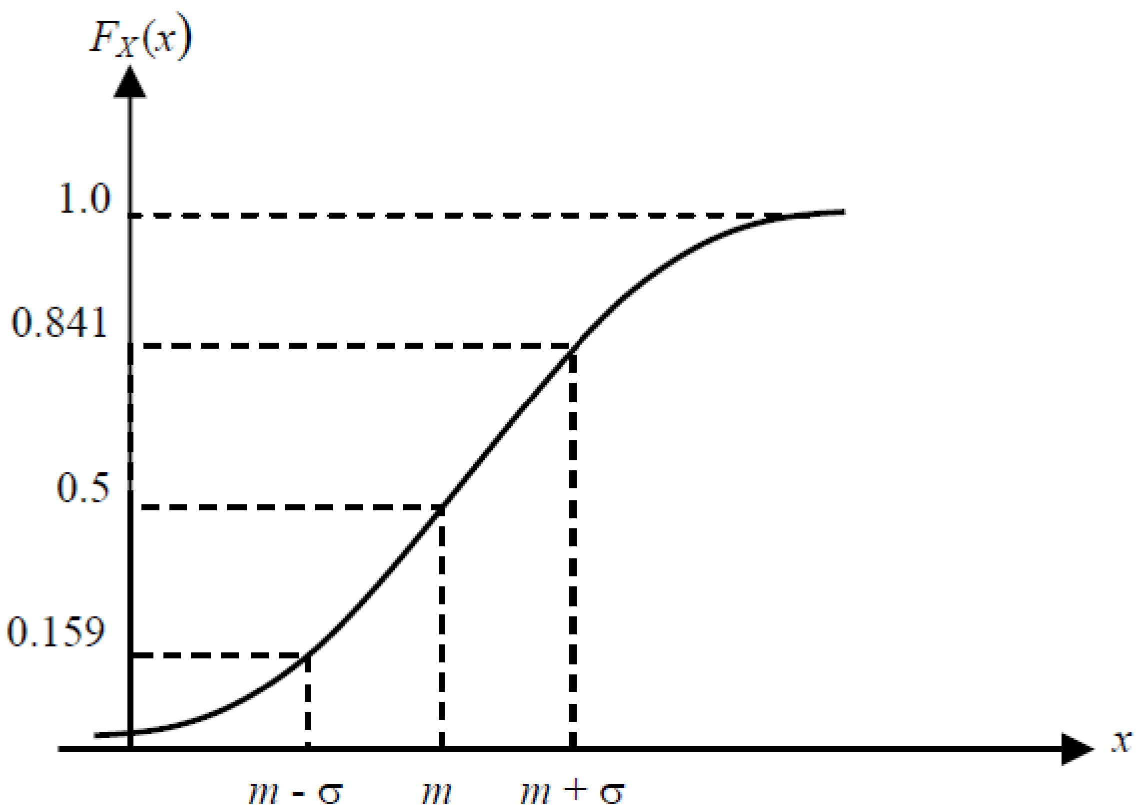
$$\sigma_x^2 = \frac{1}{12}(b-a)^2$$

Normal distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] \quad \text{for all } x$$



Normal distribution



Other distributions

- Exponential distribution
- Laplacian distribution
- Gamma distribution
- Beta distribution
- Chi-square distribution
- Etc…

Take home message

1. Moments
2. Expected value, mean and varaince
3. Two random variables
4. Joint distribution function
5. Indenpendent and uncorrelated random variables
6. Some useful distribution functions:

Exercises

Course website:

http://person.hst.aau.dk/enk/ST6

If requested:
Username: sp1
Password: st6