Welcome to Stochastic Processes I
Lecture 2: Moments and multi dimensional random variables
Expectation operator

\[ E\{g(X)\} \triangleq \int_{-\infty}^{+\infty} g(x)f_X(x) \, dx \]

Is defined only for random variables or a function of them.

**Expected value:** Is the weighted average of all possible values that this random variable can take on.

Let \( g(X) = X \)

\[ E\{X\} \triangleq \int_{-\infty}^{+\infty} xf_X(x) \, dx \]
Example 1

**Last time:** A second action potential is firing between 100 and 200 ms. We found:

\[
 f_T(t) = \begin{cases} 
 0, & t < 100 \text{ and } t > 200 \\
 \frac{1}{100}, & 100 \leq t \leq 200 
\end{cases}
\]

Thus

\[
 E\{T\} = \int_{-\infty}^{+\infty} t f_T(t) \, dt = \int_{100}^{200} t \frac{1}{100} \, dt = 150 \text{ ms}
\]
Definition of moments

A moment of order $k$ of a random variable is defined as:

$$
\mu_k = E\{X^k\} = \int_{-\infty}^{+\infty} x^k f_X(x) \, dx
$$

Order 1: $\mu_1 = E\{X\} = \int_{-\infty}^{+\infty} x f_X(x) \, dx$  Mean of X

Order 2: $\mu_2 = E\{X^2\} = \int_{-\infty}^{+\infty} x^2 f_X(x) \, dx$
Central moments

Defined in a similar way as moment

\[ m_k = E\{(X - E\{X\})^k\} = \int_{-\infty}^{+\infty} (x - \mu_1)^k f_X(x)\,dx \]

Variance of X

\[ m_2 = E\{(X - \mu_1)^2\} = \int_{-\infty}^{+\infty} (x - \mu_1)^2 f_X(x)\,dx = \sigma^2 \]

\[ m_2 = E\{(X - \mu_1)^2\} = E\{X^2\} - (E\{X\})^2 \]
Two random variables

Let $X$ and $Y$ be two random variables, we define the joint distribution function

$$F_{XY}(x, y) \triangleq P(X \leq x, Y \leq y)$$

and the joint probability density function as

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \geq 0$$
Useful Properties

- \( \lim_{x \to +\infty} F_{XY}(x, y) = F_Y(y) \)
- \( \lim_{y \to +\infty} F_{XY}(x, y) = F_X(x) \)
- \( \int_{-\infty}^{+\infty} f_{XY}(x, y)dx = f_Y(y) \)
- \( \int_{-\infty}^{+\infty} f_{XY}(x, y)dy = f_X(x) \)
- \( \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{XY}(x, y)dx\,dy = 1 \)

The probability that \( X \) lies between \( x_1 \) and \( x_2 \) and \( Y \) lies between \( y_1 \) and \( y_2 \) is

\[
P(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{XY}(x, y)dx\,dy
\]
Independent and uncorrelated random variables

Let $X$ and $Y$ be two random variables, we say that

$X$ and $Y$ are **independent** if

$$F_{XY}(x, y) = F_X(x) \cdot F_Y(y) \Rightarrow f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

$X$ and $Y$ are **uncorrelated** if

$$E\{XY\} = E\{X\} \cdot E\{Y\}$$

Indepenndence $\Rightarrow$ uncorrelation
Example 2

Consider now that 2 cells ($T_1$ and $T_2$) are firing with the same probability density function as in example 1. Calculate the probability that cell 1 discharges between 130 and 140 ms and cell 2 between 180 and 200 ms.

$$P(130 \leq T_1 \leq 140, 180 \leq T_2 \leq 200) = ?$$
**Expected value**

\[
E\{g(X, Y)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y)f_{XY}(x, y)dx\,dy
\]

\[
E\{XY\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf_{XY}(x, y)dx\,dy
\]

If \(X\) and \(Y\) are independent we get

\[
E\{XY\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf_X(x)f_Y(y)dx\,dy
\]

The correlation
\[
E\{XY\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf_X(x)f_Y(y) \, dx \, dy \\
= \int_{-\infty}^{+\infty} xf_X(x) \, dx \cdot \int_{-\infty}^{+\infty} yf_Y(y) \, dy \\
= E\{X\} \cdot E\{Y\}
\]

Uncorrelated

If \( E\{XY\} = 0 \) we say \( X \) and \( Y \) are **orthogonal**
\[ E[X+Y] = E[X] + E[Y] \]

\[ m_{10} = E[X] = m_x \quad m_{01} = E[Y] = m_y \]

\[ \mu_{20} = E[(X - m_x)^2] = \sigma_x^2 \]

\[ \mu_{02} = E[(Y - m_y)^2] = \sigma_y^2 \]

\[ \rho_{xy} = \frac{E[(X - m_x) (Y - m_y)]}{\sigma_x \sigma_y} \]

\[ C_{xy} = E[(X - m_x) (Y - m_y)] \]

Correlation coefficient

Covariance
CONTINUOUS DISTRIBUTION FUNCTIONS
The Uniform Distribution

The probability density function is given as

\[ f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases} \]

\[ F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(u) \, du = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases} \]
Uniform distribution

\[ E[X] = \frac{1}{2} (a + b) \]

\[ \sigma_x^2 = \frac{1}{12} (b - a)^2 \]
Normal distribution

\[ f_X(x) = \frac{1}{\sqrt{2\pi \sigma}} \exp \left[ -\frac{(x - m)^2}{2\sigma^2} \right] \quad \text{for all } x \]
Normal distribution
Other distributions

• Exponential distribution
• Laplacian distribution
• Gamma distribution
• Beta distribution
• Chi-square distribution
• Etc...
Take home message

1. Moments
2. Expected value, mean and variance
3. Two random variables
4. Joint distribution function
5. Independent and uncorrelated random variables
6. Some useful distribution functions:
Exercises

Course website:

http://person.hst.aau.dk/enk/ST6

If requested:
Username: sp1
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