# STOCHASTIC SIGNALS AND PROCESSES

## Lecture 2: Discrete-time random processes

## WELCOME

# Summary of Lecture I

- Definition of a stochastic/deterministic process
- Introduction to basic probability
- Random variables:
    - Continuous random variables
    - Distribution functions
        - $F_X(x) \triangleq P(X \leq x)$
        - $f_X(x) = \dfrac{dF_X(x)}{dx}$
        - $F_{XY}(x,y) \triangleq P(X \leq x, Y \leq y)$
        - $f_{XY}(x,y) = \dfrac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \quad \geq \quad 0$

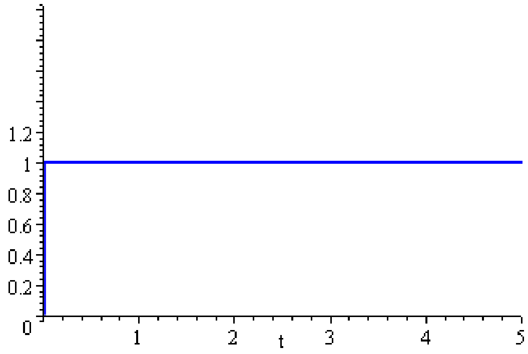
# Summary of Lecture I

- Random variables:
  - $E\{g(X)\} \triangleq \int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$
  - Mean and variance
  - Indenpendency $f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$
  - Uncorrelation $E\{XY\} = E\{X\} \cdot E\{Y\}$
  - Orthogonality $E\{XY\} = 0$
- Continuous random processes:
  - Mean and autocorrelation function
  - *Stationarity, Ergodicity, Power spectrum (Important)*

# Step and Impulse functions

A *unit (Heaviside) step function* is defined as

$$u(x) = \begin{cases} 1, x \geq 0 \\ 0, x < 0 \end{cases}$$



A *step function* of height A occurring at x=$x_0$ is denoted as defined as

$$Au(x - x_0) = \begin{cases} A, x \geq x_0 \\ 0, x < x_0 \end{cases}$$

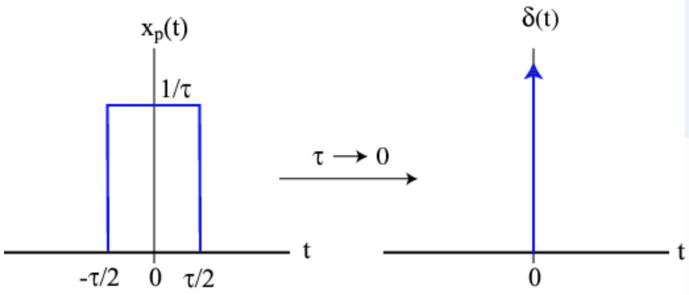
# Step and Impulse functions

The *unit impulse function* (dirac delta, an impulse with unit area) is denoted by $\delta(x)$

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \qquad \int_{-\infty}^{+\infty} \delta(x)dx = 1$$

*Useful property*

$$\int_{-\infty}^{+\infty} A\delta(x - x_0)f(x)dx = Af(x_0)$$



# Discrete random variables

Recall: A *random variable (X)* is a *real function* that maps the elements of the sample space S into points ($x$) of the real axis.

Assumes only a perticular finite or counting infinite set of values $x_1, x_2, \ldots, x_n$

To each outcome $x_i$, is associated a probability $P(x_i) = P(X=x_i)$

$P(x_i) \geq 0$ for all $i$ $\qquad$ $f_X(x) \geq 0$

and $\sum_{i=1}^{\infty} P(x_i) = 1$ $\qquad$ $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$

$$F_X(x) = P(X \leq x)$$

# Discrete-time stochastic processes

A discrete stochastic process may be a uniformly sample version of a continuous-time process.

A *collection* or an *ensemble* of real or complex discrete sequences of time, aso called realizations and denoted as $X(n)$ or $X[n]$.

# Definitions

The expected value or mean value is

$$E[X(n)] = \sum_{i=1}^{\infty} x_i p_i = m_x(n)$$

And the autocorrelation is

$$r_{xx}(n_1, n_2) = E[X(n_1)X^*(n_2)]$$

The covariance function is

$$c_{xx}(n_1, n_2\,) = E\{[X(n_1) - m_x(n_1)][X(n_2) - m_x(n_2)]^{\,*}\}$$
$$= \; r_{xx}(n_1 n_2) - m_x(n_1)m^*{}_x(n_2)$$

**WSS** if mean constant and $r_{xx}(n, n+k) = r_{xx}(k)$

# Definitions

$$r_{xx}(0) \geq |r_{xx}(k)| \; and \; r_{xx}(-k) = r^{*}{}_{xx}\,(k)$$

Power spectral density: provides important informations about the structure of the random process

$$S_{xx}(\omega) = \sum_{k=-\infty}^{\infty} r_{xx}(-k)e^{-j\omega k}$$

White noise

$$S_{xx}(\omega) = \sigma^2$$

Flat power spectrum

The ***mean square value*** = *average power* in the discrete-time random process

$$r_{xx}(0) = \mathrm{E}[|\mathrm{X}(\mathrm{n})|^2] = \int_{-\frac{f}{2}}^{\frac{f}{2}} S_{xx}(f)df$$

# Discrete linear models

1. Autoregressive processes
2. Moving average processes

Two stationary linear models often used to model random sequences.

Can be derived from data

Most used empirical models of random sequences

# Discrete linear models

Their combinations describe the output of a LLTIVC system

**Lumped**: A dynamic system is called lumped if it can be modeled by a set of ordinary difference equations.

**Linear:** $f\big(ax(t)+by(t)\big)=af\big(x(t)\big)+bf(y(t))$

**Time invariant:**

If $y(t)=f\big(x(t)\big)$, then $y(t-t_0)=f\big(x(t-t_0)\big)$

**Causal:** output depends on pass input, not future.

$$y(t_0)=f[x(t);t\leq t_0]$$

# Autoregressive process (AR model)

An autoregressive process is one represented by a difference equation of the form:

$$X(n) = \sum_{k=1}^{p} -a_k X(n-k) + e(n)$$

$a_k$ are parameters

$p$ is the model order

$e(n)$ sequence of iid zero mean gaussian random variable or *white gaussian noise*, that is:

$$\mathrm{E}[e(n)] = 0, \qquad \mathrm{E}[e(n)^2] = \sigma_n^2$$

# Autoregressive process (AR model)

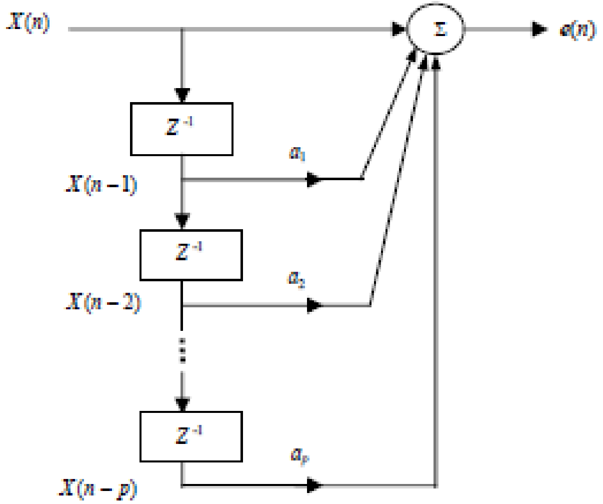
$$X(n) = \sum_{k=1}^{p} -a_k X(n-k) + e(n)$$

The transfer function of of the all-zero filter is

$$H(z) = \frac{E(Z)}{X(Z)}$$

$$= 1 + \sum_{k=1}^{p} a_k Z^{-k}$$

Signal whitening



***<u>Assumptions:</u>***

1)The process *X(n)* is stationary, *e(n)* is stationary.

*2)X(0)* is zero-mean Gaussian.

*3)X(0)* is independent of *e(n)*.

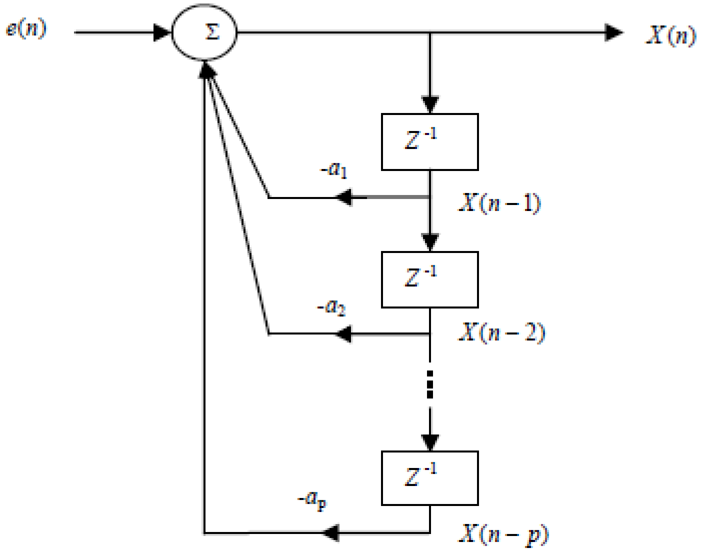
# Autoregressive process (AR model)

However when the input is white noise e(*n*)

$$H(z) = \frac{X(Z)}{E(Z)} = \frac{1}{1+\sum_{k=1}^{p} a_k Z^{-k}}$$

The corresponding *AR(p)* filter is all-pole.

*Thus An autoregressive model can be viewed as the output of an all-pole infinite impulse response filter whose input is white noise.*



OR
A linear difference equation model when the input is gaussin noise

# First order AR model (*AR(1)*)

$$X(n) = -a_1 X(n-1) + e(n)$$

**Mean:** $\mu_X = \mathrm{E}[X(n)] = 0, \quad \text{for} \quad a_1 \neq -1$

**Variance:** $\sigma_X^2 = E[X(n)^2] = \dfrac{\sigma_n^2}{1-a_1^2} \qquad -1 < a_1 < 1$

**Autocorrelation function:** $r_{XX}(k) = E[X(n)X(n-k)] = (-a_1)^k r_{XX}(0) = (-a_1)^k \sigma_X^2$

**Power spectrum:**

# First order AR model (*AR(1)*)

$$X(n) = -a_1 X(n-1) + e(n)$$

**Transfer function**:

$$H(f) = \frac{1}{1 + a_1 e^{-j2\pi f}}, \quad |f| < \frac{1}{2}$$

**Power spectrum**:

$$S_{XX}(f) = |H(f)|^2 S_{ee}(f)$$

$$= \frac{\sigma_n^2}{1 + 2a_1 \cos 2\pi f + a_1^2}, \quad |f| < \frac{1}{2}$$

# p^th order AR model (*AR(p)*)

$$X(n) = -\sum_{k=1}^{p} a_k X(n-k) + e(n)$$

**Mean:** $\mu_X = E[X(n)] = 0$ for $\sum_{k=1}^{p} a_k \neq -1$

**Autocorrelation function:**

$$r_{XX}(l) = \begin{cases} -\sum_{k=1}^{p} a_k r_{XX}(l-k), & l > 0 \\ -\sum_{k=1}^{p} a_k r_{XX}(k) + \sigma_n^2, & l = 0 \end{cases}$$

**Variance:** $\sigma_X^2 = E[X(n)^2] = r_{XX}(0) = -a_k \sum_{k=1}^{p} r_{XX}(k) + \sigma_n^2$

# $p^{th}$ order AR model (*AR(p)*)

$$X(n) = -\sum_{k=1}^{p} a_k X(n-k) + e(n)$$

**Power spectrum:** $S_{XX}(f) = |H(f)|^2 S_{ee}(f), \qquad f < \left|\frac{1}{2}\right|$

$$= \frac{\sigma_n^2}{\left|1 + \sum_{k=1}^{p} a_k e^{-j2\pi f.k}\right|^2}$$

# p<sup>th</sup> order AR model (*AR(p)*)

**Autocorrelation function:**

$$r_{XX}(l) = \begin{cases} -\sum_{k=1}^{p} a_k r_{XX}(l-k), & l > 0 \\ -\sum_{k=1}^{p} a_k r_{XX}(k) + \sigma_n^2, & l = 0 \end{cases}$$

$$\boxed{\sigma_n^2} = r_{XX}(0) + \sum_{k=1}^{p} a_k r_{XX}(k)$$

$$\underbrace{\begin{bmatrix} r(1) \\ r(2) \\ \vdots \\ r(p) \end{bmatrix}}_{\mathbf{r}} = \underbrace{\begin{bmatrix} r(0) & r(1) & r(2) & \cdots & r(p-1) \\ r(1) & r(0) & r(1) & \cdots & r(p-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r(p-1) & r(p-2) & r(p-3) & \cdots & r(0) \end{bmatrix}}_{\mathbf{R}} \underbrace{\boxed{\begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_p \end{bmatrix}}}_{\mathbf{a}}$$

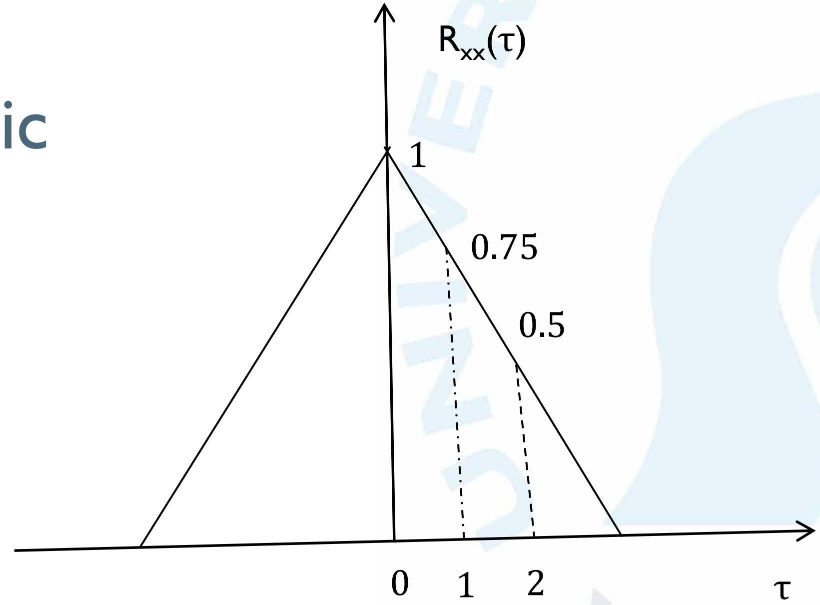
**Yule-Walker equations**

$$\mathbf{r} = \mathbf{R}\mathbf{a} \qquad \mathbf{a} = \mathbf{R}^{-1}\mathbf{r}$$

**R**: Hermitian toeplitz matrix

# Example

Given the following autocorrelation function of a stochastic process X(n), fit the data to AR(2) model.



# Moving average process (MA model)

$$X(n) = \sum_{k=0}^{q} b_k e(n-k)$$

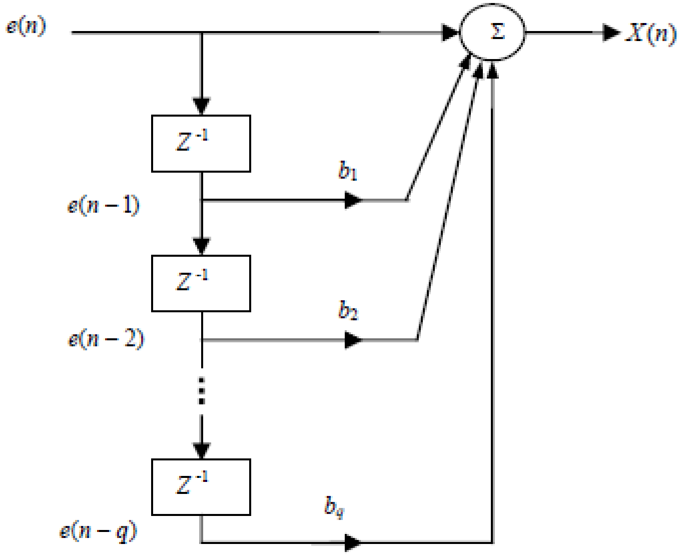
$$\sum_{k=0}^{q} b_k = 1$$

or

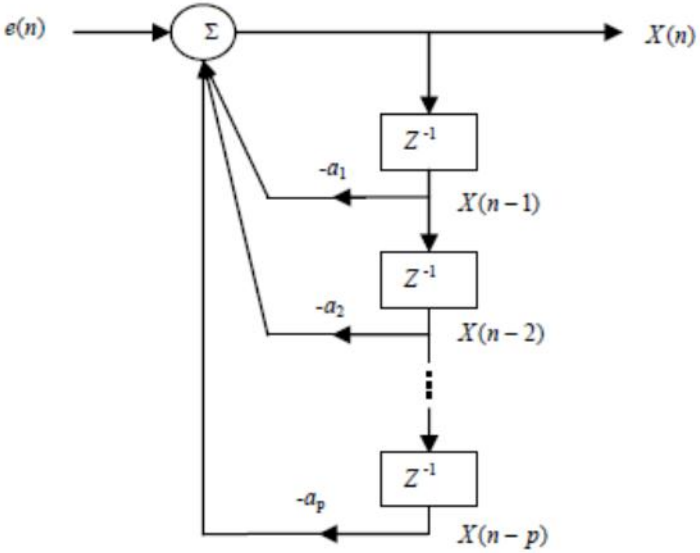
$$X(n) = \sum_{k=1}^{q} b_k e(n-k) + e(n)$$

where $b_0 = 1, \quad b_k \neq 0 \text{ for } k = 1,2,\ldots,q.$

# Moving average process (MA model)



MA model



AR model

# First order MA model (*MA(1)*)

$$X(n) = b_1 e(n-1) + e(n)$$

**Mean:** $$\mu_X = \mathrm{E}\left[X(n)\right] = \mathrm{E}\left[b_1 e(n-1) + e(n)\right] = 0$$

**Variance:** $$\sigma_X^2 = \mathrm{E}\left[X(n)^2\right] = (b_1^2 + 1)\sigma_n^2$$

**Autocorrelation function:** $$r_{XX}(k) = \mathrm{E}\left[X(n)X(n-k)\right]$$

$$= \begin{cases} b_1\sigma_n^2, & k = 1 \\ 0, & k > 1 \end{cases}$$

# First order MA model (*MA(1)*)

**Transfer function**:

$$H(f) = 1 + b_1 e^{-j2\pi f}, \quad |f| < \frac{1}{2}$$

**Power spectrum**:

$$S_{XX}(f) = |H(f)|^2 S_{ee}(f)$$

$$= \sigma_n^2 \left|1 + b_1 e^{-j2\pi f}\right|^2 = \sigma_n^2 (1 + 2b_1 \cos 2\pi f + b_1^2), \quad |f| < \frac{1}{2}$$

# $q^{th}$ order MA model ($MA(q)$)

$$X(n) = \sum_{k=1}^{q} b_k e(n-k) + e(n)$$

**Mean:** $$m_X = \mathrm{E}[X(n)] = \sum_{k=1}^{q} \mathrm{E}\left[b_k e(n-k) + e(n)\right] = 0$$

**Variance:** $$\sigma_X^2 = E[X(n)^2] = \sigma_n^2 \left(1 + \sum_{k=1}^{q} b_k^2\right)$$

**Autocorrelation function:** $$R_{XX}(k) = \begin{cases} \sigma_n^2 \left(b_k + \sum_{j=k+1}^{q} b_j b_{j-k}\right), & k < q \\ \sigma_n^2 b_q, & k = q \\ 0, & k > q \end{cases}$$

# $q^{th}$ order MA model ($MA(q)$)

$$X(n)=\sum_{k=1}^{q} b_k e(n-k)+e(n)$$

**Power spectrum:**

$$S_{XX}(f)=\left|H(f)\right|^2 S_{ee}(f), \qquad |f|<\frac{1}{2}$$

$$=\sigma_n^2\left|1+\sum_{k=1}^{q} b_k e^{-j2\pi f\cdot k}\right|^2$$

# Autoregressive moving average process (ARMA model)

**ARMA(p, q):**

$$X(n)=\underbrace{\sum_{k=1}^{p}-a_k X(n-k)}_{\text{AR part}}+e(n)+\underbrace{\sum_{l=1}^{q}b_l e(n-l)}_{\text{MA part}}$$

A combination of both models

# Next lecture

An application oriented lecture
With focus on power spectrum estimation

How to estimate the power spectral density using the autocorrelation, the signal itself, AR and ARMA models

Examples of biosignals