Stochastic Processes

Lecture 2

Exercise 1

Consider the random process

$X(n) = \frac{1}{2} X(n-1) + Z(n), n = 0,1,...$

Where Z(n) is stationary white Gaussian noise with zero mean and $\sigma_Z^2 = 1$. X(0) is a Gaussian random variable, which is independent of Z(n), N≥1.Find the mean and variance of X(0) in order for the process to be stationary.

Solution:

$$E\{X(n)\} = \frac{1}{2} E\{X(n-1)\} + E\{Z(n)\} = \frac{1}{2} E\{X(n-1)\}$$

If X(n) is stationary

$$E\{X(n)\} = E\{X(n-1)\} = \mu_x$$

$$\mu_x = \frac{1}{2}\mu_x$$

$$\mu_x = 0$$

Variance:

$$Var\{X(n)\} = E\{X^2(n)\} = E\left\{\frac{1}{4}X^2(n-1) + Z^2(n) + X(n-1)\cdot Z(n)\right\} =$$

$$E\left\{\frac{1}{4}X^2(n-1)\right\} + E\{Z^2(n)\} + E\{X(n-1)\cdot Z(n)\} =$$

$$\frac{1}{4}\sigma^2_{X(n-1)} + \sigma_Z^2 + 0 = \frac{1}{4}\sigma^2_{X(n-1)} + \sigma_Z^2$$

If X(n) is stationary $\sigma^2_{X(n)} = \sigma^2_{X(n-1)} = \sigma^2_X$

$$\sigma^2_{X(n)} = \frac{1}{4}\sigma^2_{X(n-1)} + \sigma_Z^2$$

$$\sigma_X^2 = \frac{4}{3}$$

Exercise 2

Given the following samples of a stochastic process X(n), fit the data to AR(2) model.

K = -4	0.9705
K = -3	1.0055
K = -2	0.9318
K = -1	0.9472
K = 0	1.0549
K = +1	0.8952
K = +2	1.0646
K = +3	1.0280
K = +4	0.9609

Solution:

The estimated the autocorrelation are:

T = -2	0.7722
T = -1	0.8889
T = 0	1.0000
T = 1	0.8889
T = 2	0.7722

Using the Yule Walker equations:

$$\begin{bmatrix} r(1) \\ r(2) \end{bmatrix} = \begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix} \begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix}$$

$$\begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix} = \begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix}^{-1} \begin{bmatrix} r(1) \\ r(2) \end{bmatrix} = \begin{bmatrix} 1 & 0.8889 \\ 0.8889 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.8889 \\ 0.7722 \end{bmatrix} = \begin{bmatrix} 0.9649 \\ -0.0855 \end{bmatrix}$$

Thus the AR(2) model is $X(n) = 0.9646X(n-1) - 0.0855X(n-2) + e(n)$

$\sigma_n^2 = E\left[e(n)^2\right] = r(0) + a_1 r(1) + a_2 r(2) = 0.2083$

Exercise 3 (optional)

For ARMA(1,1) process X(n), find for the process (a) μ_X (b) σ_X^2 (c) $R_{XX}(k)$ (d) $S_{XX}(f)$.

Solution:

The ARMA(1,1) process can be described by $X(n) = -a_1 X(n-1) + b_1 e(n-1) + e(n)$

Mean:

$$\mu_X = \mathrm{E}[X(n)] = \mathrm{E}[-a_1 X(n-1) + b_1 e(n-1) + e(n)] = -a_1 \mu_X$$

Thus $\mu_X = 0$, if $a_1 \neq -1$

Variance:

$$\sigma_X^2 = \mathrm{E}[X(n)^2] = \mathrm{E}\{[-a_1 X(n-1) + b_1 e(n-1) + e(n)] \cdot [-a_1 X(n-1) + b_1 e(n-1) + e(n)]\}$$
$$= \mathrm{E}\left\{a_1^2 X(n-1)^2 - 2a_1 b_1 X(n-1)e(n-1) - 2a_1 X(n-1)e(n) + 2b_1 e(n-1)e(n) + b_1^2 e(n-1)^2 + e(n)^2\right\}$$
$$= \mathrm{E}\left\{a_1^2 X(n-1)^2 - 2a_1 b_1 X(n-1)e(n-1) + b_1^2 e(n-1)^2 + e(n)^2\right\}$$
$$= \mathrm{E}\left\{a_1^2 X(n)^2 - 2a_1 b_1 X(n)e(n) + b_1^2 e(n)^2 + e(n)^2\right\}$$
$$= a_1^2 \sigma_X^2 - 2a_1 b_1 \sigma_n^2 + (1 + b_1^2)\sigma_n^2$$

Thus $\sigma_X^2 = \dfrac{(1 + b_1^2 - 2a_1 b_1)\sigma_n^2}{(1 - a_1^2)}, \qquad -1 < a_1 < 1$

Autocorrelation:

$$R_{XX}(k) = \mathrm{E}[X(n-k)X(n)]$$
$$= \mathrm{E}[-a_1 X(n-k)X(n-1) + b_1 X(n-k)e(n-1) + X(n-k)e(n)]$$
$$= -a_1 \mathrm{E}[X(n-k)X(n-1)] + b_1 \mathrm{E}[X(n-k)e(n-1)] + \mathrm{E}[X(n-k)e(n)]$$
$$= \begin{cases} -a_1 R_{XX}(0) + b_1 \sigma_n^2, & k = 1 \\ -a_1 R_{XX}(k), & k \geq 2 \end{cases}$$

Power spectrum:

$$S_{XX}(f) = |H(f)|^2 S_{ee}(f) = \frac{\sigma_n^2 \left|1 + b_1 e^{-j2\pi f}\right|^2}{\left|1 + a_1 e^{-j2\pi f}\right|^2}, \qquad |f| < \frac{1}{2}$$