STOCHASTIC PROCESSES II

Solutions for lecture 3

Exercise 1

A system for home recording of ECG signals transmits the processed signals to a local hospital using a simple amplitude modulation. The transmitted signal X contains only the information on the sequence of heart beats:

$$X = \begin{cases} +A, & QRS \Rightarrow H_1 \\ 0, & No \quad QRS \Rightarrow H_0 \end{cases}$$

Where A>0.

At the receiver we have

Y=X+W

Where W is a zero-mean Gaussian random process with $\sigma^2 = \frac{A^2}{4}$. The receiver has a simple threshold algorithm that decides if the transmitted sample correspond to a QRS or not and estimate the instantaneous cardiac frequency.

1) Find the probability density functions of Y conditioned to H_0 and H_1.

The expected value under H1 is E[Y] = +A and the variance is the same as $\sigma^2 = \frac{A^2}{4}$

$$f(y \mid H_1) = \frac{2}{\sqrt{2\pi}A} e^{-\frac{2}{A^2}(y-A)^2}$$

$$f(y \mid H_0) = \frac{2}{\sqrt{2\pi}A} e^{-\frac{2}{A^2}y^2}$$

2) Calculate the log-likelihood ratio $\ln\Lambda(y) = \ln\left(\frac{f(y \mid H_1)}{f(y \mid H_0)}\right)$

$$l(y) = \ln\left(e^{\frac{2}{A^2}\left[-(y-A)^2+y^2\right]}\right) = \frac{2}{A^2}\left[-y^2 - A^2 + 2Ay + y^2\right] = -2 + \frac{4y}{A}$$

Exercise 3

Consider the hypothesis testing problem in which

$$f_{Y|H_1}(y \mid H_1) = \frac{1}{2}\,\mathrm{rect}\left(\frac{y-1}{2}\right) \quad \text{and} \quad f_{Y|H_0}(y \mid H_0) = e^{-y} \ \text{for} \ y > 0$$

a) Set up the likelihood ratio test and determine the decision regions.
b) Find the minimum probability of error when $P_0 = 2/3$

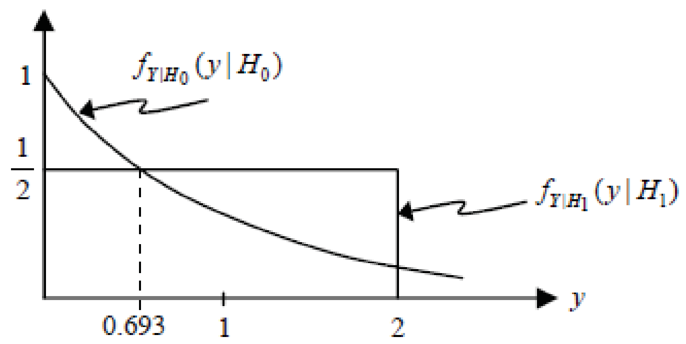
Hint:

$$\mathrm{rect}(t) = \sqcap(t) = \begin{cases} 0 & \text{if } |t| > \frac{1}{2} \\ \frac{1}{2} & \text{if } |t| = \frac{1}{2} \\ 1 & \text{if } |t| < \frac{1}{2}. \end{cases}$$

Solutions

a)

The LRT is $\Lambda(y) = \dfrac{f_{Y|H_1}(y \mid H_1)}{f_{Y|H_0}(y \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$



We observe that for $0 \le y \le 2 \Rightarrow \dfrac{1/2}{e^{-y}} \underset{H_2}{\overset{H_1}{\gtrless}} \eta \Rightarrow y \underset{H_2}{\overset{H_1}{\gtrless}} \ln(2\eta)$, while for $y > 2$, we always decide H_0.

b)

We now know that we should find η for minimum probability of error. For this condition the costs are given as $C_{00} = C_{11} = 0$ and $C_{01} = C_{10} = 1$, thus $\eta = P_0/P_1$.

For $P_0 = 2/3 \Rightarrow P_1 = 1/3$ and $\eta = 2$. The minimum probability of error is

$$P(\varepsilon) = P_1 \int_{1.39}^{2} e^{-y}\,dy + P_0 \int_{0}^{1.39} \frac{1}{2}\,dy = 0.308$$

Exercise 4

The max of s(t) is:

$$s_{mat}^{\max}(t) = \int_{-T}^{T} h_{mat}(t)\cdot s(t)dt = \int_{-T}^{T} s^2(t)dt = E_s = 2\int_{0}^{1}(-t+1)^2\,dt = \frac{2}{3}$$

$$Noise:$$

$$N_{mat}(t) = N(t) * h_{mat}(t)$$

$$\mu_{mat} = H(0)\cdot\mu_N = 0$$

$$\sigma_{mat}^2 = \int_{-\infty}^{+\infty} S_{NN}(f)\cdot|H(f)|^2\,df = \frac{1}{10}\int_{-\infty}^{+\infty} h^2(t)dt = \frac{1}{10}E_s = \frac{1}{15}$$

The probability density functions for the two hypotheses are:

H_1 :

$$f_{y|H_1}(y\,|\,H_1) = \frac{1}{\sqrt{2\pi}\sigma_{mat}} e^{-\frac{(y-s_{mat}^{\max})^2}{2\sigma_{mat}^2}} = \frac{1}{\sqrt{\frac{2\pi}{15}}} e^{-\frac{(y-\frac{2}{3})^2}{2/15}}$$

H_0 :

$$f_{y|H_0}(y\,|\,H_0) = \frac{1}{\sqrt{2\pi}\sigma_{mat}} e^{-\frac{y^2}{2\sigma_{mat}^2}} = \frac{1}{\sqrt{\frac{2\pi}{15}}} e^{-\frac{(y)^2}{2/15}}$$

The Likelihood ratio test is:

$$l(y)=\frac{f_{y|H_1}(y\,|\,H_1)}{f_{y|H0}(y\,|\,H_0)}$$

$$\ln(l(y)) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{P(H_0)}{P(H_1)}\right)$$

$$-(y-\frac{2}{3})^2+y^2 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{2}{15}\ln(19)$$

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{3}{4}\frac{2}{15}\ln(19)+\frac{1}{3}=0.63=\lambda$$

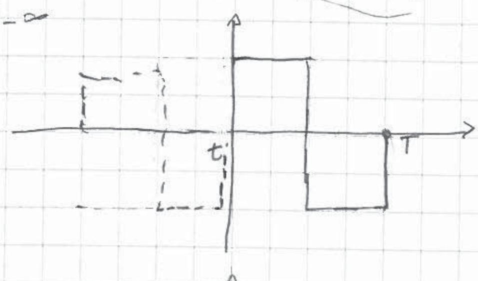
Exercise 2

$$y_1(t) = S_1(t) * h(t)$$

$$= \int_{-\infty}^{\infty} S_1(u) \cdot S(T-t-u)\,du$$

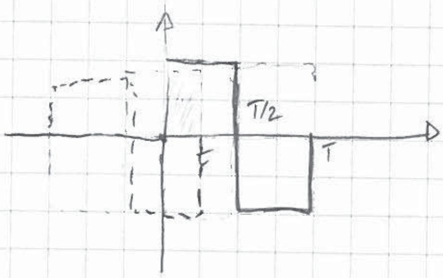
⇒ For $t < 0$

$S_1(t) = 0$



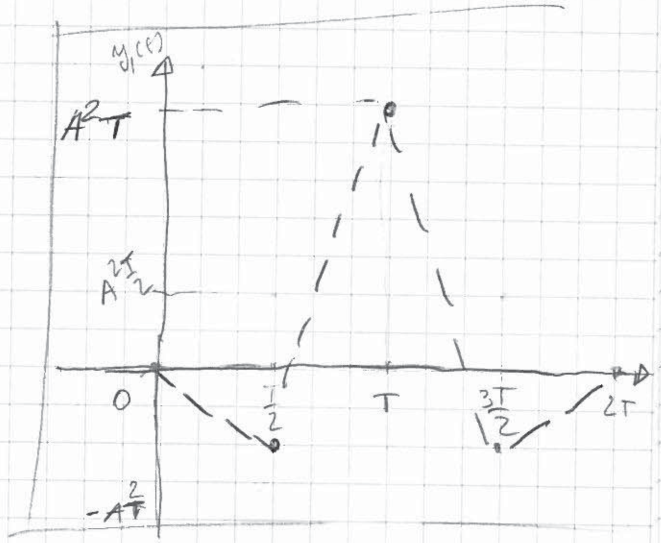
$y_1(t) = 0$

⇒ For $0 \leq t < \frac{T}{2}$

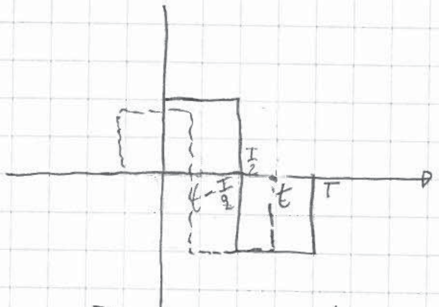


$$y_1(t) = \int_0^t -A \cdot A\,du = -A^2 t$$

$S_1(t) = A$
$h_1(t) = A$



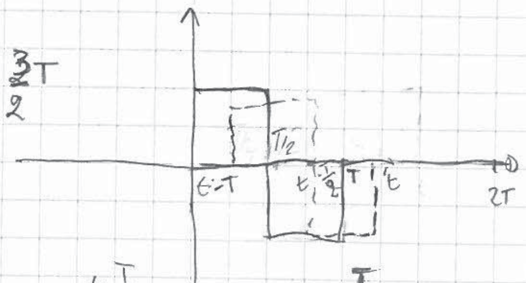
⇒ For $\frac{T}{2} \leq t \leq T$



$$y_1(t) = \int_0^{t-\frac{T}{2}} +A^2\,du + \int_{t-\frac{T}{2}}^{\frac{T}{2}} -A^2\,du + \int_{\frac{T}{2}}^{t} +A^2\,du$$

$$= +A^2\left(t-\frac{T}{2}\right) - A^2\left(\frac{T}{2}-t+\frac{T}{2}\right) + A^2\left(t-\frac{T}{2}\right) = -A^2(2T-3t)$$

⇒ For $T \leq t < \frac{3}{2}T$

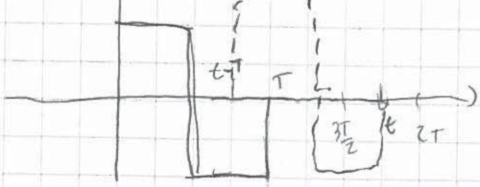


$$y_1(t) = \int_{t-T}^{\frac{T}{2}} +A^2\,du + \int_{\frac{T}{2}}^{t-\frac{T}{2}} -A^2\,du + \int_{t-\frac{T}{2}}^{T} +A^2\,du$$

$$= +A^2\left(\frac{T}{2}-t+T\right) - A^2\left(t-\frac{T}{2}-\frac{T}{2}\right) + A^2\left(T-t+\frac{T}{2}\right)$$

$$= -A^2(3t-4T)$$

For $\frac{3}{2}T \leq t < 2T$



$$y_1(t) = \int_{t-T}^{T} -A^2\,du = -A^2(2T-t)$$

$y_1(t) = 0$ For $t \geq 2T$