STOCHASTIC PROCESSES II

Solutions for Lecture 4

Exercise 1

We will investigate a very simple detector which is based on a threshold set on the estimate of the mean value of the signal. Consider the signal $X[n]$ represented below:

\[
\begin{align*}
X[n] & \quad \text{A} \\
\text{T} & \quad \text{n [ms]}
\end{align*}
\]

with $A=1$ and $T=10 \text{ ms}$, $F_c=1000 \text{ Hz}$. The received signal can be modeled as following

\[
Y(n) = X(n) + W(n)
\]

Suppose that the noise $W[n]$ is white, Gaussian, with zero mean and $\sigma_w^2 = 1$. Demonstrate that

\[
\bar{Y} = \frac{1}{N} \sum_n Y(n)
\]

has $\mu_Y = 1$ and in case of no event $\sigma_Y^2 = \frac{\sigma_w^2}{N}$.

Solution

The mean can be easily shown that is equal to 1.

The variance of the estimator is:
Exercise 2

In this exercise, we will investigate the use of linear prediction for building a compression methods applied to EMG signals.
Consider the stochastic process \( S[n] \) that models an EMG signal recording. \( S[n] \) is zero mean and has autocorrelation function \( R_{xx}[m] \), represented below

\[
E\{\bar{Y} - \mu_y\}^2 = E\{\bar{Y} - \mu_x\}^2 = E\left\{ \frac{1}{N} \sum_k (X(k) + W(k)) - \mu_x \right\}^2 = \\
E\left\{ \frac{1}{N} \sum_k (X(k) + W(k))^2 \right\} + \mu_x^2 - 2\mu_x \frac{1}{N} \sum_k (X(k) + W(k)) = \\
\frac{1}{N^2} \sum_n \sum_m E\{X(n) + W(n)(X(m) + W(m))\} + \mu_x^2 - 2\mu_x \frac{1}{N} \sum_k E\{X(k) + W(k)\} = \\
\frac{1}{N^2} \sum_n \sum_m \left( E\{X(n)X(m)\} + E\{X(n)W(m)\} + E\{W(n)X(m)\} + E\{W(n)W(m)\} \right) + \\
+ \mu_x^2 - 2\mu_x \frac{1}{N} \sum_k E\{X(k) + W(k)\} = \\
\frac{1}{N^2} \sum_n \sum_m \left( \mu_x^2 + E\{W(n)W(m)\} \right) + \mu_x^2 - 2\mu_x \frac{1}{N} \sum_k E\{X(k)\} = \\
\frac{1}{N^2} N^2 \mu_x^2 + \frac{N}{N^2} \sigma_N^2 + \mu_x^2 - 2\mu_x \frac{1}{N} N \mu_x = \frac{\sigma_N^2}{N}
\]
\[ \hat{X}[k] = h_0 \cdot X[k - 2] + h_1 \cdot X[k - 1] \]

Using Minimization of the MSE we have:

For \( i = k-2 \Rightarrow h_0 \cdot R_{xx}(k-2-k+2) + h_1 \cdot R_{xx}(k-1-k+2) = R_{xx}(k-k+2) \)

For \( i = k-1 \Rightarrow h_0 \cdot R_{xx}(k-2-k+1) + h_1 \cdot R_{xx}(k-1-k+1) = R_{xx}(k-k+1) \)

Solving the system with equations

\[
\begin{align*}
&h_0 \cdot R_{xx}(0) + h_1 \cdot R_{xx}(1) = R_{xx}(2) \\
&h_0 \cdot R_{xx}(1) + h_1 \cdot R_{xx}(0) = R_{xx}(1)
\end{align*}
\]

we have

\[
\begin{align*}
h_0 &= -0.05256 \\
h_1 &= 0.9474
\end{align*}
\]

2) Find the mean and variance of the error when using the linear predictor developed in point 1.

\[ mean \]

\[
E\left[ \left( X[k] - \hat{X}[k] \right) \right] = E\left[ (X[k] - h_1 \cdot X[k - 1] - h_2 \cdot X[k - 2]) \right] = 0
\]

\[ variance \]

\[
E\left[ \left( X[k] - \hat{X}[k] \right)^2 \right] = E\left[ (X[k] - h_1 \cdot X[k - 1] - h_2 \cdot X[k - 2])^2 \right] =
\]

\[
E\left[ X^2[k] + h_1^2 X^2[k-1] + h_2^2 X^2[k-2] - 2h_1 X[k] X[k-1] - 2h_2 X[k] X[k-2] + 2h_1 h_2 X[k-1] X[k-2] \right] =
\]

\[
R_{xx}[0] + h_1^2 R_{xx}[0] + h_2^2 R_{xx}[0] - 2h_1 R_{xx}[1] - 2h_2 R_{xx}[2] + 2h_1 h_2 R_{xx}[1] = 0.1895
\]