Solutions for Lecture 5

8.1 (*a*) We have, $\int_0^T \left(\frac{1}{\sqrt{T}} \sqrt{\frac{2}{T}} \cos \frac{k\pi t}{T} \right) dt = 0$

$$\int_0^T \frac{1}{\sqrt{T}} \frac{1}{\sqrt{T}} dt = 1$$

and

$$\int_0^T \left(\sqrt{\frac{2}{T}} \cos \frac{k\pi t}{T} \sqrt{\frac{2}{T}} \cos \frac{j\pi t}{T} \right) dt = \begin{cases} 1 & , \quad k = j \\ 0 & , \quad k \neq j \end{cases}$$

Therefore, $\left\{ \frac{1}{\sqrt{T}}, \sqrt{\frac{2}{T}} \cos \frac{k\pi t}{T} \right\}$ are orthonormal functions.

(*b*) Similarly, to verify that the set functions is orthonormal in the interval $[-1, 1]$, we do $\int_{-T}^{T} \frac{1}{\sqrt{2T}} \frac{1}{\sqrt{2T}} dt = 1$

$$\int_{-T}^{T} \frac{1}{\sqrt{2T}} \frac{1}{\sqrt{T}} \cos \frac{k\pi t}{T} dt = 2\int_0^T \frac{1}{T\sqrt{2}} \cos \frac{k\pi t}{T} dt = 0$$

and

$$\int_{-T}^{T} \frac{1}{\sqrt{T}} \cos \frac{k\pi t}{T} \frac{1}{\sqrt{T}} \cos \frac{j\pi t}{T} dt = \frac{2}{T} \int_0^T \cos \frac{k\pi t}{T} \cos \frac{j\pi t}{T} dt = \delta_{kj}$$

Hence, the set is orthonormal on the interval $[-1, 1]$.

8.2 (a) We solve $\int_{-1}^{1} s_1(t)s_2(t)dt = \int_{-1}^{1} tdt = 0$

$$\int_{-1}^{1} s_1^2(t)dt = 2\int_{0}^{1} dt = 2$$

and

$$\int_{-1}^{1} s_2^2(t)dt = \int_{-1}^{1} t^2 dt = 2\int_{0}^{1} t^2 dt \frac{2}{3}$$

Therefore, $s_1(t)$ and $s_2(t)$ are orthogonal.

(b) $s_1(t)$ orthogonal to $s_3(t) \Rightarrow \int_{-1}^{1} 1(1+\alpha t+\beta t^2)dt = 0 \Rightarrow \beta = -3$

$s_2(t)$ orthogonal to $s_3(t) \Rightarrow \int_{-1}^{1} t(1+\alpha t+\beta t^2)dt = 0 \Rightarrow \alpha = 0$.

Therefore, $s_3(t) = 1-3t^2$.

8.3 Note that $s_3(t) = -2s_1(t) \Rightarrow$ We have 2 independent signals.

The energy of $s_1(t)$ is thus,

$$E_1 = \int_{0}^{T} s_1^2(t)dt = \int_{0}^{T/2} 1dt + \int_{T/2}^{T} (-1)^2 dt = \frac{T}{2} + \frac{T}{2} = T$$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E}} = \begin{cases} \frac{1}{\sqrt{T}} & , \quad 0 \le t \le \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & , \quad \frac{T}{2} \le t \le T \end{cases}$$

$f_2(t) = s_2(t) - s_{21}\phi_1(t)$ where $s_{21} = \int_0^T s_2(t)\phi_1(t)dt$. Then,

$$s_{21} = \int_0^{T/2} (-1)\left(\frac{1}{\sqrt{T}}\right)dt + \int_{T/2}^{T} (-2)\left(-\frac{1}{\sqrt{T}}\right)dt = +\frac{\sqrt{T}}{2}$$

$$f_2(t) = s_2(t) - s_{21}\phi_1(t) = -\frac{3}{2} \qquad 0 \le t \le T$$

and

$$\phi_2(t) = \frac{-3/2}{\sqrt{\int_0^T \left(-\frac{3}{2}\right)^2 dt}} = -\frac{1}{\sqrt{T}} \qquad 0 \le t \le T$$

(b) $s_1(t) = \sqrt{T}\phi_1(t)$

$$s_2(t) = \frac{\sqrt{T}}{2}\phi_1(t) + \frac{3}{2}\sqrt{T}\phi_2(t)$$

$$s_3(t) = -2\sqrt{T}\phi_1(t)$$

Thus, the signal constellation is