AALBORG UNIVERSITY

Welcome to Time-Frequency Analysis, Adaptive Filtering and Source Separation

Lecture 5: Discrete Wavelet Transform MultiResolution Analysis

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Power spectra

- Classical methods
- Modern approaches
  - Parametric methods
  - Non parametric
- Time Frequency analysis
  - Time-frequency methods
  - Time-scale methods (**WAVELET**)

From surface to deep learning

Storyline

Questions and Answers

Each group will come up with 2 questions on CWT.
I will pick one member to read the questions for the audience. Each group will work on the answers during the exercises time.

Wavelet transform

- None of the methods developed so far has solved the time-frequency problem without constraints or added artifacts.

- Fourier transform
- $F(m) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j2\pi mt} dt$

- A sine wave is perfectly localized in frequency, but infinite in time.

Short-time fourier transform

$$STFT(t,f) = \int_{-\infty}^{+\infty} x(\tau) w(t-\tau) e^{-j2\pi f t} d\tau$$

In general

$$X(t,m) = \int_{-\infty}^{+\infty} x(\tau) \cdot f(t-\tau)_m d\tau$$

In the wavelet transform (WT), family members consist of enlarged and compressed versions of the basic function

Continuous Wavelet Transform (CWT)

From french: ondelette (small wave)

- Finite in time
- $W(a,b) = \int_{-\infty}^{+\infty} x(t) \cdot \frac{1}{\sqrt{a}} \psi^{*}\left(\frac{t-b}{a}\right) \mathrm{dt}$
- Different values of *a* and *b* gives a serie of wavelets that may be addedd together to reconstruct the signal
- ***They are all localized in both time and frequency, but not precisely localized in either.***

Continuous Wavelet Transform (CWT)

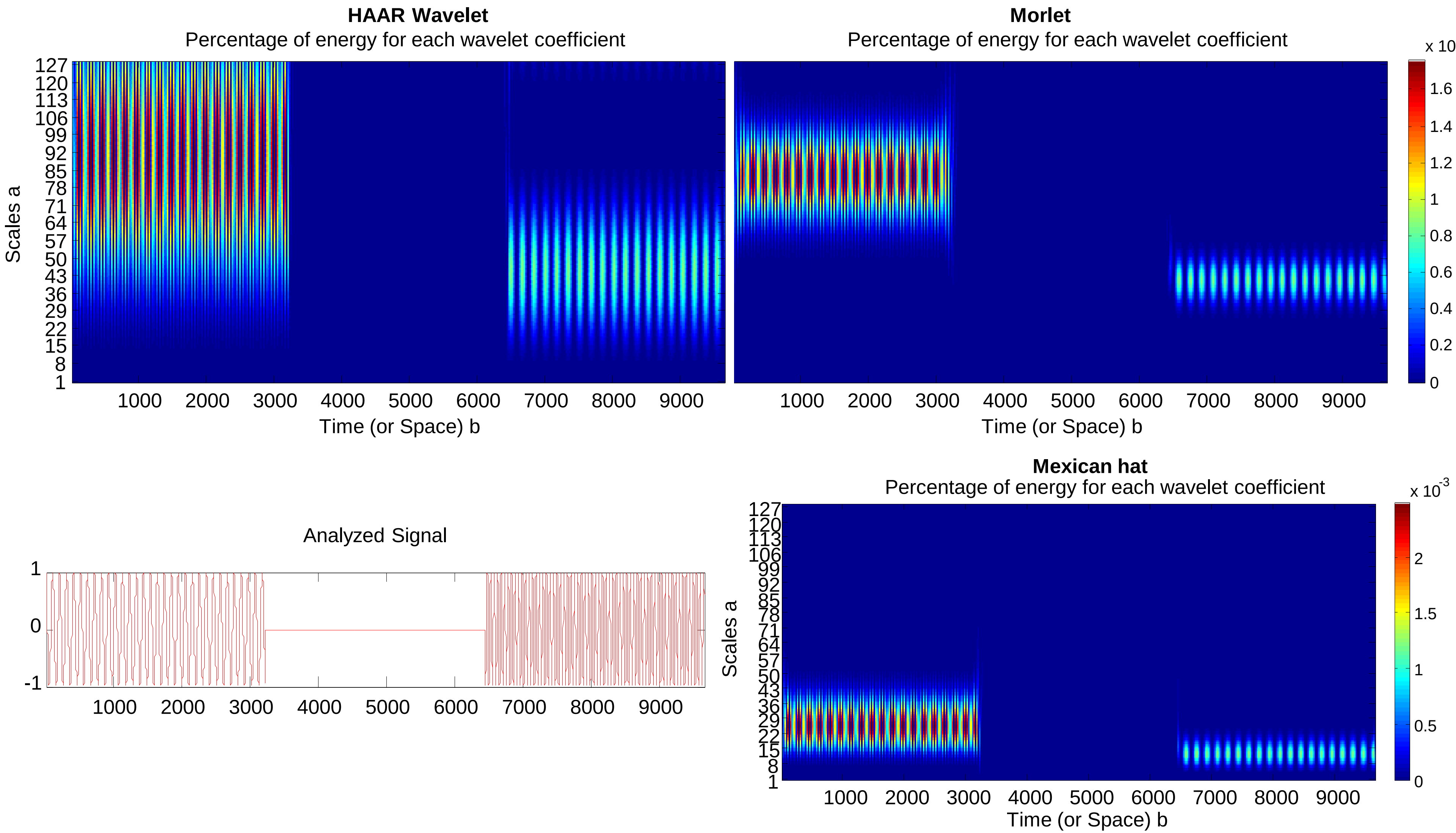
CWT

- $W(a,b) = \int_{-\infty}^{+\infty} x(t) \cdot \frac{1}{\sqrt{a}} \psi^* \left(\frac{t-b}{a}\right) \mathrm{dt}$

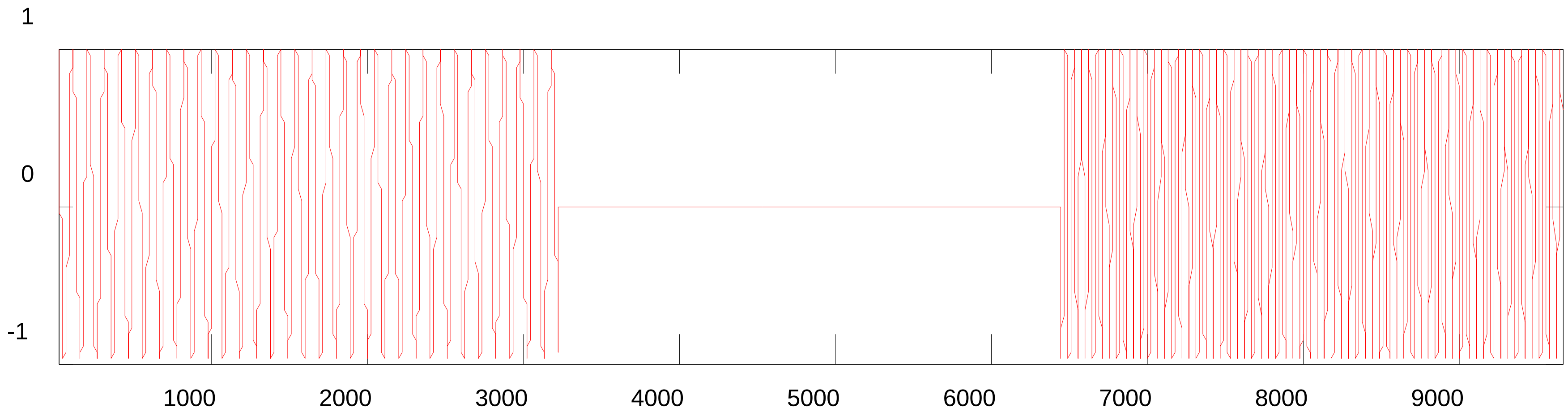
iCWT

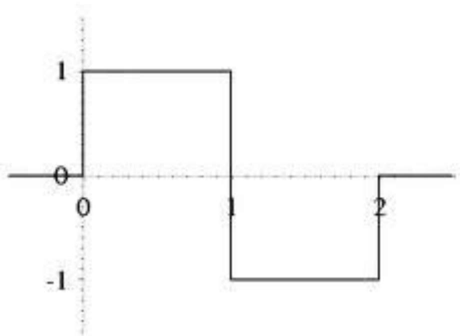
- $x(t) = \frac{1}{C} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W(a,b) \cdot \psi^*(t) \mathrm{d}a db$

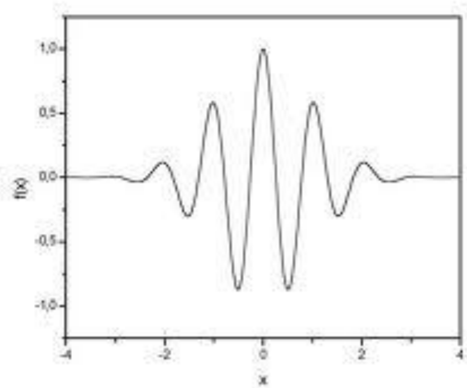
Solutions

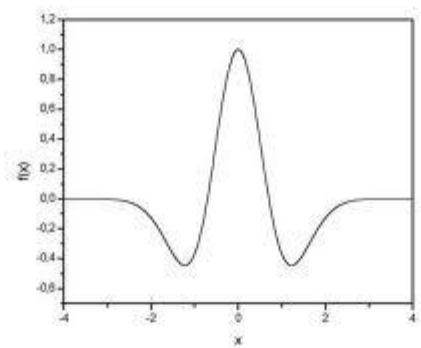

HAAR Wavelet
Percentage of energy for each wavelet coefficient
Scales a
127 120 113 106 99 92 85 78 71 64 57 50 43 36 29 22 15 8 1
1000 2000 3000 4000 5000 6000 7000 8000 9000
Time (or Space) b
Morlet
Percentage of energy for each wavelet coefficient
x 10
1.6 1.4 1.2 1 0.8 0.6 0.4 0.2 0
1000 2000 3000 4000 5000 6000 7000 8000 9000
Time (or Space) b
Analyzed Signal
1 0 -1
1000 2000 3000 4000 5000 6000 7000 8000 9000
Mexican hat
Percentage of energy for each wavelet coefficient
x 10^-3
2 1.5 1 0.5 0
Scales a
127 120 113 106 99 92 85 78 71 64 57 50 43 36 29 22 15 8 1
1000 2000 3000 4000 5000 6000 7000 8000 9000
Time (or Space) b

Solutions


1
0
-1
1000
2000
3000
4000
5000
6000
7000
8000
9000


1
0
0
1
2
-1


1,0
0,5
0,0
-0,5
-1,0
f(x)
-4
-2
0
2
4
x

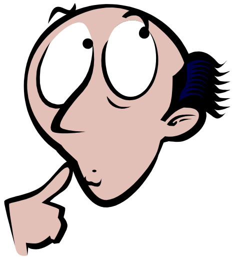

1,2
1,0
0,8
0,6
0,4
0,2
0,0
-0,2
-0,4
-0,6
f(x)
-4
-2
0
2
4
x

Discrete Wavelet Transform (DWT)

From CWT → DWT

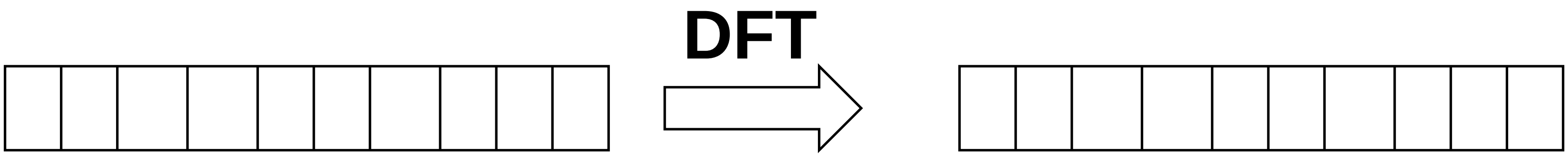
Why *CWT* and *DWT*?

But we also talk about Continous fourier transform (CFT) and discrete FT.



Discrete Fourier Tranform

DFT comes from CFT, but computed for some specific values:



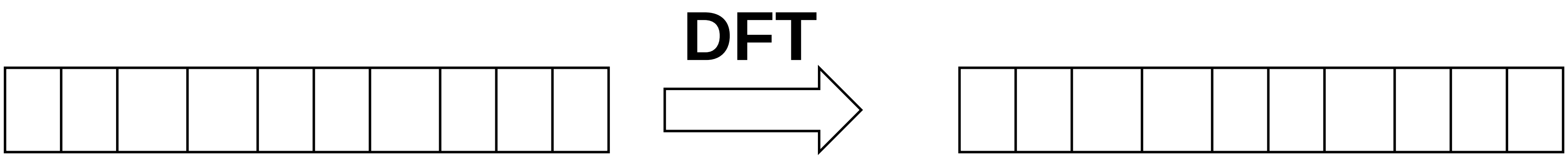
x(t) with N samples *X(f) with N samples*

Thus same number of samples at the output: **<u>NON REDUDANT</u>**

CFT can be computed for any f value

$$X(f) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j2\pi f t} dt$$

Discrete Fourier Tranform



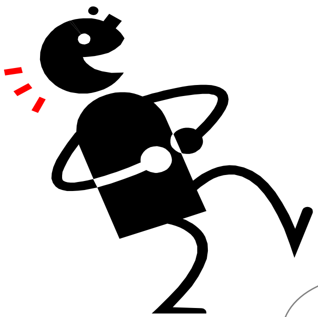
x(t) with N samples   *X(f) with N samples*

In **DFT** we choose only ***N*** frequency values that are equispaced so that the transformation is invertible.

DFT = special version of CFT that is non redudant and invertible

Discrete Wavelet Transform (DWT)

DFT and CFT



Why *CWT* and *DWT*?

CWT: Any value of *a* and *b*.
DWT: **Choose only values so that <u>non redudant and invetible</u>.**
DFT: 1-D, but DWT: 2-D

Series expansions

$$f(t) = \sum_{i=-\infty}^{+\infty} \alpha_i \cdot \phi_i(t)$$

The set of functions $\{\phi_i(t)\}$, with $i \in R$, are bases for $f(t)$.

$$f(t) \in L^2$$

Orthonormality: Orthogonal and unitary energy

$$< \phi_i(t), \phi_j(t) > = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

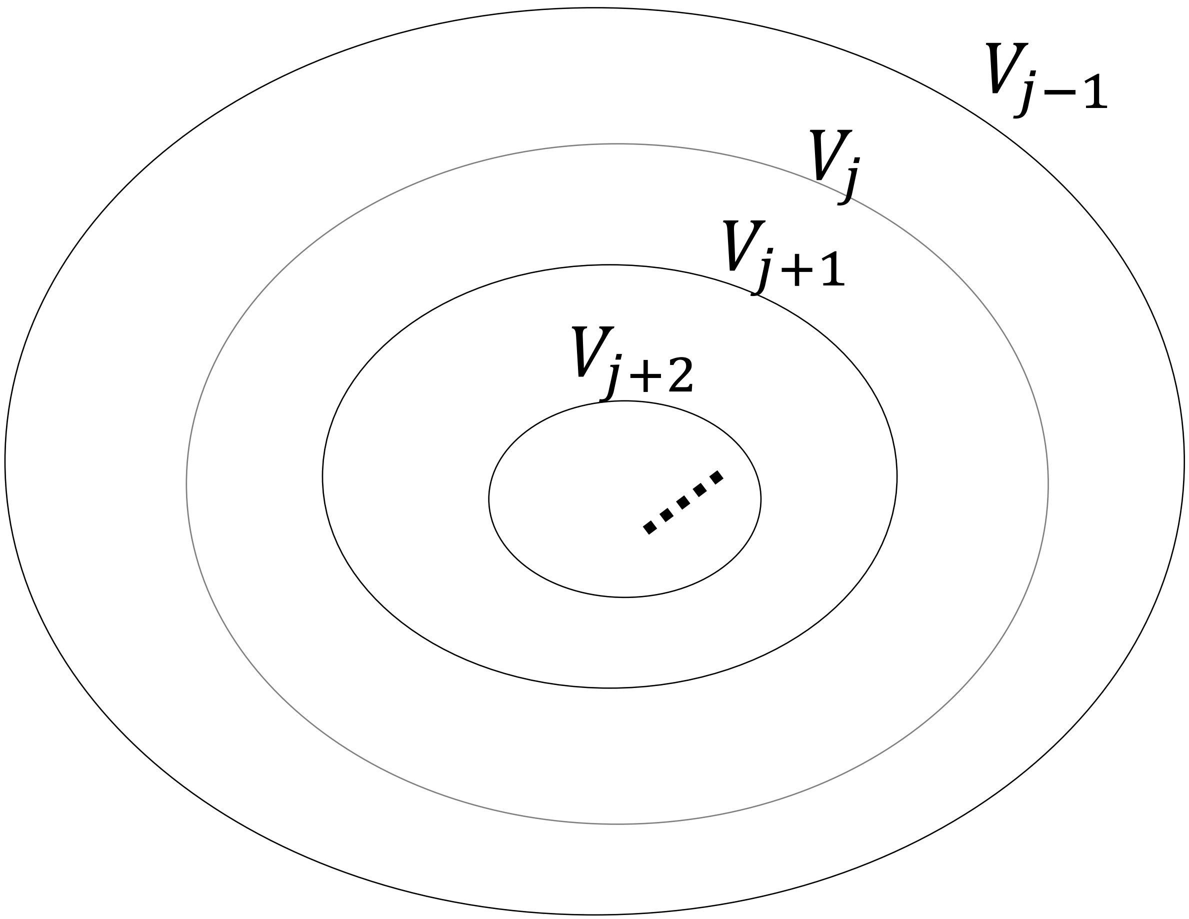
Series expansions

$$f(t) = \sum_{i=-\infty}^{+\infty} \alpha_i \cdot \phi_i(t)$$

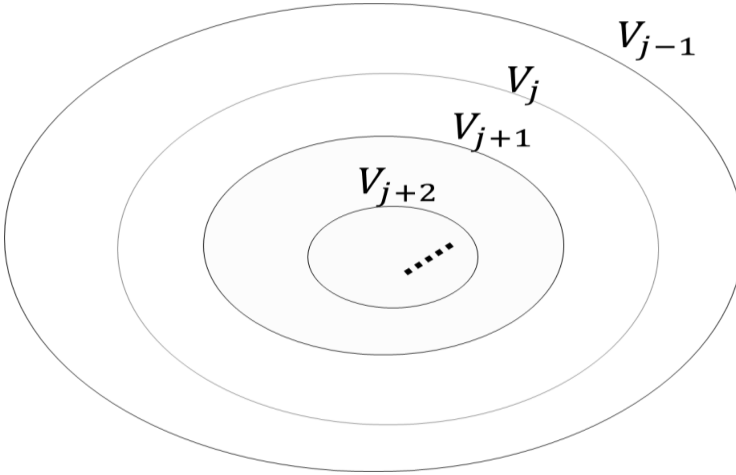
How to compute α_i ?

$$\alpha_j =< f(t), \phi_j(t) >$$

MultiResolution Analysis (MRA)



MRA properties



1) $\ldots \subset V_{j+2} \subset V_{j+1} \subset V_j \subset V_{j-1} \subset \cdots$

2) $\lim_{j\to+\infty} V_j = \{\emptyset\}$

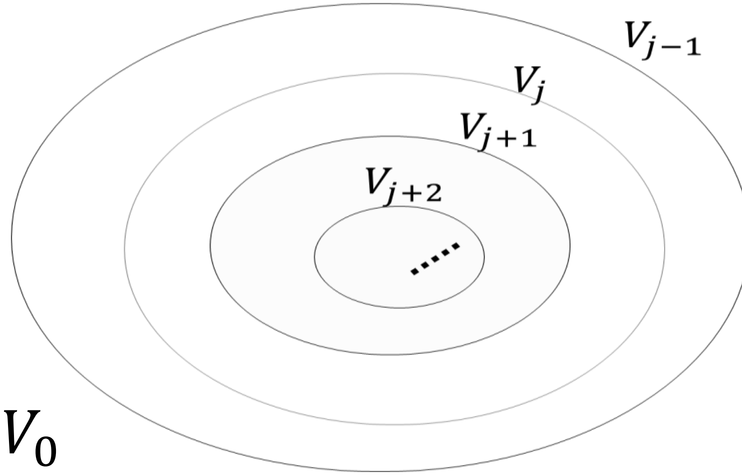
3) $\lim_{j\to-\infty} V_j = L^2$

4) If $f(t) \in V_0$ then $f(t-n) \in V_0 \quad for\, n \in R$

5) If $f(t) \in V_j$ then $f\left(\frac{t}{2}\right) \in V_{j+1}$

6) $\{\theta(t-n)\}$ are orthonormal bases for V_0

MRA properties



5) If $f(t) \in V_j$ then $f\left(\frac{t}{2}\right) \in V_{j+1}$

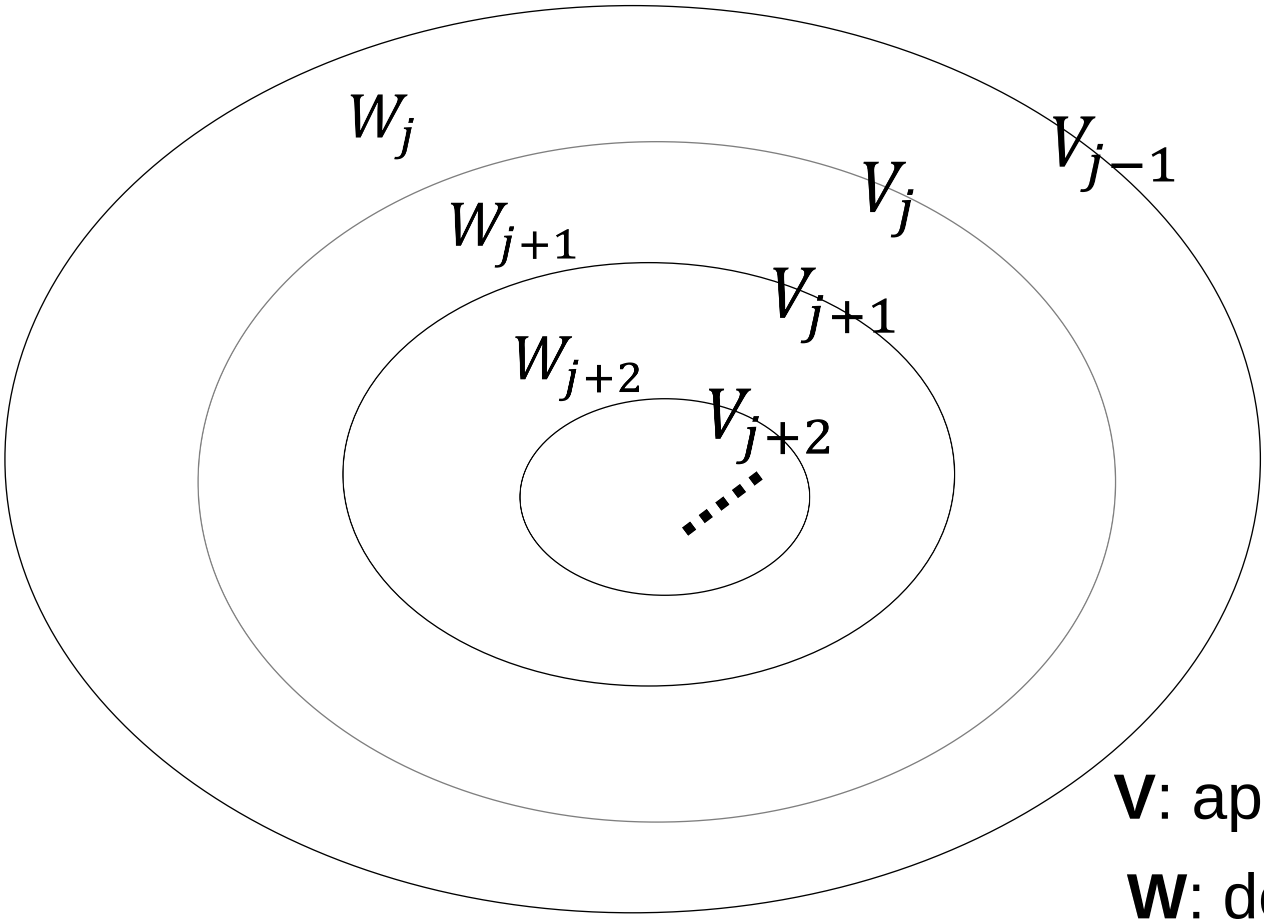
6) $\{\theta(t-n)\}$ are orthonormal bases for V_0

Combining properties (5) and (6) we have

$$\left\{2^{-\frac{j}{2}} \cdot \theta\left(2^{-j}t-n\right)\right\} \ as\ orthonormal\ bases\ for\ V_j$$

$\theta(t)$ is called Scaling function

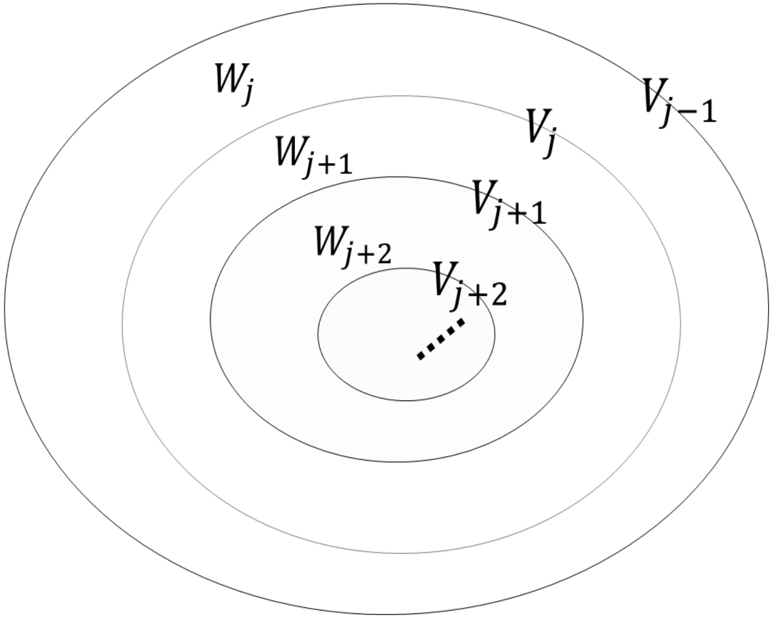
Approximation and detail spaces



V: approximation space

W: detail space

Approximation and detail spaces



$$
\begin{aligned}
V_{j-1} &= V_j \oplus W_j \\
&= V_{j+1} \oplus W_{j+1} \oplus W_j \\
&= V_{j+2} \oplus W_{j+2} \oplus W_{j+1} \oplus W_j
\end{aligned}
$$

$$V_{j-1} = \bigcup_{k=0}^{+\infty} W_{j+k}$$

Thus $L^2 = \bigcup_{j=-\infty}^{+\infty} W_j$

$$\left\{2^{-\frac{j}{2}} \cdot \psi(2^{-j}t - n)\right\} \text{ as orthonormal bases for } W_j$$

$\psi(t)$ is called wavelet function

Two scale equations

Relationship between 2 scales

$$\theta(t) = \sum_{n=-\infty}^{+\infty} h_n \cdot \sqrt{2}\,\theta(2t-n)$$

$$\psi(t) = \sum_{n=-\infty}^{+\infty} g_n \cdot \sqrt{2}\,\theta(2t-n)$$

$$< \theta(t-n), \psi(t-m) > = 0, Orthogonal$$

Example: MRA of *f(t)*

Exercises

Questions and Answers

Part I: Each group will come up with 2 questions on CWT.
I will pick one member to read the questions for the audience. Each project will work on the answers during the exercises time.

What did you not understand?

Part II: Each group will come up with 2 questions on DWT.
(30 minutes)

Group Work (45 minutes)