Welcome to Time-Frequency Analysis, Adaptive Filtering and Source Separation

Lecture 6: Filter Banks
Wavelet Packet and Parameterization

Ernest N. Kamavuako
From surface to deep learning

Storyline

Questions and Answers
Continuous Wavelet Transform (CWT)

From french: ondelette (small wave)

- Finite in time

\[ W(a, b) = \int_{-\infty}^{+\infty} x(t) \cdot \frac{1}{\sqrt{a}} \psi^* \left( \frac{t-b}{a} \right) dt \]

- Different values of \( a \) and \( b \) gives a serie of wavelets that may be added together to reconstruct the signal

- They are all localized in both time and frequency, but not precisely localized in either.
Continuous Wavelet Transform (CWT)

CWT

\[ W(a, b) = \int_{-\infty}^{+\infty} x(t) \cdot \frac{1}{\sqrt{a}} \psi^* \left( \frac{t-b}{a} \right) dt \]

iCWT

\[ x(t) = \frac{1}{c} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W(a, b) \cdot \psi^*(t) dadb \]
Discrete Wavelet Transform (DWT)

DFT and CFT

Why CWT and DWT?

CWT: Any value of $a$ and $b$. 
DWT: Choose only values so that non-redundant and invertible.
DFT: 1-D, but DWT: 2-D
Multiresolution Analysis

\[ \theta(t) = \sum_{n=-\infty}^{+\infty} h_n \cdot \sqrt{2} \theta(2t - n) \]

\[ \psi(t) = \sum_{n=-\infty}^{+\infty} g_n \cdot \sqrt{2} \theta(2t - n) \]

\[ V_j \]: approximation space

\[ W_j \]: detail space

\[ V_{j+1} \]

\[ W_{j+1} \]

\[ V_{j+2} \]

\[ W_{j+2} \]

\[ \ldots \]
Multiresolution Analysis

\[ \left\{ 2^{-\frac{j}{2}} \cdot \theta\left(2^{-j} t - n\right) \right\} \text{ as orthonormal bases for } V_j. \]

\( \theta(t) \) is called Scaling function

\[ V_{j-1} = \bigcup_{k=0}^{+\infty} W_{j+k} \]

\[ \left\{ 2^{-\frac{j}{2}} \cdot \psi\left(2^{-j} t - n\right) \right\} \text{ as orthonormal bases for } W_j. \]

\( \psi(t) \) is called wavelet function
Discrete Wavelet transform

\[ \beta_{n,j} = \int_{-\infty}^{+\infty} f(t) \cdot \frac{1}{\sqrt{2^j}} \psi^* \left( \frac{t - 2^j n}{2^j} \right) dt \]

\[ W(a, b) = \int_{-\infty}^{+\infty} f(t) \cdot \frac{1}{\sqrt{a}} \psi^* \left( \frac{t - b}{a} \right) dt \]

\( a = 2^j \) and \( b = 2^j n \) : Dyadic wavelet transform
Filter Banks

- A filter bank is an array of band-pass filters that separates the input signal into multiple components, each one carrying a single frequency subband of the original signal.

- We have seen that multiresolution Analysis allows us to decompose a signal into approximations and details.

- Filter Bank is a way to implement the MRA and DWT.
Filter Banks

\[ P_{V_0}f = \sum_{n} c_n \theta(t - n) = f(t) \]

\[ P_{V_1}f = \sum_{k} a_k \frac{1}{\sqrt{2}} \theta \left( \frac{t}{2} - n \right) \]

\[ P_{W_1}f = \sum_{k} d_k \frac{1}{\sqrt{2}} \Psi \left( \frac{t}{2} - n \right) \]

We would like to find \( a_k \) and \( d_k \), not by using \( f(t) \) but its representation in \( V_0(c_n) \). \( a_K, d_K \)?
Analysis: from fine scale to coarser scale

Matlab functions: \textit{dwt} and \textit{wavedec}

\[
\begin{align*}
[C, C] &= \text{dwt}(x, \text{Lo}, \text{Hi}); \\
&= \text{dwt}(x, \text{'wname'});
\end{align*}
\]

\[
\begin{align*}
[C, L] &= \text{wavedec}(x, N, \text{Lo}, \text{Hi}); \\
&= \text{wavedec}(x, N, \text{'wname'});
\end{align*}
\]
Analysis: from fine scale to coarser scale
Max scale

Input

$\alpha = 2^2$

$\alpha = 2^3$

$\alpha = 2^4$

$\alpha = 2^5$

Output

Time (samples)
Synthesis: from coarse scale to fine scale

\[ d_1[n] \rightarrow \uparrow 2 \rightarrow g[n] \rightarrow + \rightarrow C[n] \]

\[ a_1[n] \rightarrow \uparrow 2 \rightarrow h[n] \]

Matlab functions: \textit{idwt} and \textit{waverec}

\[ x = \textit{idwt}(cA, cD, \text{Lo, Hi}); \]
\[ x = \textit{idwt}(cA, cD, \text{'wname'}); \]
\[ x = \textit{waverec}(C, L, \text{Lo, Hi}); \]
\[ x = \textit{waverec}(C,L, \text{'wname'}); \]
Wavelet Packet
Wavelet Parameterization

- WT requires the selection of the mother wavelet.
- Wavelet usually designed similar to the signal.
- Here The mother wavelet is parameterized.
- \( \psi \) is defined by a low-pass filter \( h \) and its associated high-pass filter \( g \).

\[
\begin{align*}
  h[0] &= (1 - \cos(\alpha) + \sin(\alpha)) / (2\sqrt{2}) \\
  h[1] &= (1 + \cos(\alpha) + \sin(\alpha)) / (2\sqrt{2}) \\
  h[2] &= (1 + \cos(\alpha) - \sin(\alpha)) / (2\sqrt{2}) \\
  h[3] &= (1 - \cos(\alpha) - \sin(\alpha)) / (2\sqrt{2})
\end{align*}
\]
Wavelet Parameterization

\[ g[n] = (-1)^{1-n} h[1 - n] \]

If \( \alpha = 0 \), \( h = \left[ 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right] \) \( \Rightarrow \) \( g = \left[ 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right] \)

\([h,g] = \text{wfilters('db2')}\) \textbf{Flip h and change signs of odd values}

\[
\begin{align*}
h[0] &= (1 - \cos(\alpha) + \sin(\alpha))/(2\sqrt{2}) \\
h[1] &= (1 + \cos(\alpha) + \sin(\alpha))/(2\sqrt{2}) \\
h[2] &= (1 + \cos(\alpha) - \sin(\alpha))/(2\sqrt{2}) \\
h[3] &= (1 - \cos(\alpha) - \sin(\alpha))/(2\sqrt{2})
\end{align*}
\]

\[
\begin{align*}
h &= [-0.1294, 0.2241, 0.8365, 0.4830], \\
g &= [-0.4830, 0.8365, -0.2241, -0.1294]
\end{align*}
\]