Lecture 7
Summary and Applications of Joint Time-Frequency Analysis

Time-frequency analysis, adaptive filtering and source separation

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Overview

• **Summary of JTFA**
  - Short Time Fourier Transform
  - Wigner-Ville distribution
  - Kernel properties and design in Cohen’s class time-frequency distributions

• **Biomedical applications of JTFA**
What the Fourier transform misses...

\[ x(t) \]

\[ y(t) \]

Single-Sided Amplitude Spectrum of \( x(t) \)

Single-Sided Amplitude Spectrum of \( y(t) \)
Timing is also important!

• For many signals, it is not enough to know the global frequency content.

• We also need to know the timing in which these changes in frequency occur, in order to follow the dynamics of the signal.

• Which signals are these?
  
  • Non-stationary, transient, whose parameters change with time (derived from a non-LTI system).
Some toy examples...
... And some real examples
... And some real examples
... And some real examples
Analyze by segments using the FT
Short-Time Fourier Transform (STFT)

- Basic approach: slicing the waveform of interest into a number of short segments and performing the Fourier transform on each one of them.

- A window function is applied to a segment of the signal, thus isolating it from the overall waveform.

\[ s_t(\tau) = s(\tau)h(\tau - t) \]
Short-Time Fourier Transform (STFT)

- Since the modified signal emphasizes the original signal around the time \( t \), the Fourier transform will reflect the distribution of frequencies around that time

\[
S_t(\omega) = \frac{1}{\sqrt{2\pi}} \int s_t(\tau)e^{-j\omega\tau}d\tau
\]

\[
S_t(\omega) = \frac{1}{\sqrt{2\pi}} \int s(\tau)h(\tau - t)e^{-j\omega\tau}d\tau
\]
Short-Time Fourier Transform (STFT)
Short-Time Fourier Transform (STFT)

• The energy density spectrum at time $t$ is

$$P_{SP}(t, \omega) = |S_t(\omega)|^2 = \left| \frac{1}{\sqrt{2\pi}} \int s(\tau) h(\tau - t) e^{-j\omega \tau} d\tau \right|^2$$

• For each different time we get a different spectrum, and the totality of these spectra is the time-frequency distribution $P_{SP}$.

• The most common name for this distribution is spectrogram.
Spectrogram
Spectrogram

- Another way to look at it is as a change of the function basis
Uncertainty principle
Uncertainty principle

- We cannot simultaneously know time and frequency aspects of a signal at an arbitrary resolution.

- For each value of $t$ and $\omega$, there is a rectangle whose sides are determined by $\sigma_t$ and $\sigma_\omega$, and whose area is at least $\frac{1}{2}$.

- When the window $h(t)$ is selected, the resolution is fixed in time and frequency.

- Since the window $h(t)$ is always equal and just shifts in time, the STFT has an uniform resolution both in time and frequency.
Uncertainty principle
Uncertainty principle
Time-Frequency representations

- The Short-Time Fourier Transform (STFT) takes a linear approach for a time-frequency representation.
- It decomposes the signal on elementary components, called atoms.

\[ h_{t,\omega}(\tau) = h(\tau - t)e^{-j\omega\tau} \]

- Each atom is obtained from the window \( h(t) \) by a translation in time and a translation in frequency (modulation).
Time-Frequency representations

• If we consider the square modulus of the STFT, we get the **spectrogram**, which is the spectral energy density of the locally windowed signal $s_t(\tau) = s(\tau)h(\tau - t)$.

• The spectrogram is a **quadratic** or **bilinear** representation.

• If the energy of the windows is selected to be one, the energy of the spectrogram is equal to the energy of the signal.

• Thus, it can be interpreted as a measure of the energy of the signal contained in the time-frequency domain centered on the point $(t, \omega)$. 
Time-Frequency representations

• **Linear**
  • STFT
  • Wavelet

• **Bilinear or Quadratic**
  • Cohen’s class
    • Spectrogram
    • Wigner-Ville
    • Choi-Williams
    • ...
  • Affine distributions
The Wigner-Ville Distribution

- The Wigner-Ville (and all of Cohen’s class of distribution) uses a variation of the autocorrelation function where time remains in the result, called **instantaneous autocorrelation function**

\[ R_{ss}(t, \tau) = s(t + \frac{\tau}{2})s^*(t - \frac{\tau}{2}) \]

Where \( \tau \) is the time lag and \( * \) represents the complex conjugate of the signal \( s \).
The Wigner-Ville Distribution

- Instantaneous autocorrelation of four cycle sine plots
The Wigner-Ville Distribution

- The Wigner-Ville Distribution (WVD) is defined as

\[ W_s(t, \omega) = \frac{1}{2\pi} \int s(t + \tau/2)s^*(t - \tau/2)e^{-j\omega\tau} d\tau \]

or equivalently

\[ W_s(t, \omega) = \frac{1}{2\pi} \int S(\omega + \theta/2)S^*(t - \theta/2)e^{-j\theta t} d\theta \]
The Wigner-Ville Distribution
The Wigner-Ville Distribution

• In an analogy to the STFT, the window is basically a shifted version of the same signal

• It is obtained by comparing the information of the signal with its own information at other times and frequencies

• It possesses several interesting properties!
Properties of the WVD

- Energy conservation
- Real-valued
- Marginal properties
- Translation and dilation covariance
- Compatibility with filterings
- Wide-sense support conservation
- Unitarity
Interference in the WVD

• As the WVD is a bilinear function of the signal $s$, the quadratic superposition principle applies

$$W_{s+x}(t, \omega) = W_s(t, \omega) + W_x(t, \omega) + 2\Re\{W_{s,x}(t, \omega)\}$$

where

$$W_{s,x}(t, \omega) = \frac{1}{2\pi} \int s(t + \tau/2)x^*(t - \tau/2)e^{-j\omega\tau} d\tau$$

is the cross-WVD of $s$ and $x$
Interference in the WVD
Interference in the WVD
Interference in the WVD
Interference in the WVD

• These interference terms are troublesome since they may overlap with auto-terms (signal terms) and thus make it difficult to visually interpret the WVD image.

• It appears that these terms must be present or the good properties of the WVD (marginal properties, instantaneous frequency and group delay, localization, unitarity . . . ) cannot be satisfied

• There is a trade-off between the quantity of interferences and the number of good properties
Pseudo-WVD

- The definition of the WVD requires the knowledge of

\[ q_s(t, \omega) = s(t + \tau/2)s^*(t - \tau/2) \]

from \( t = -\infty \) to \( t = +\infty \), which can be a problem in practice

- Often a windowed version of \( q_s(t, \omega) \) is used, leading to the Pseudo-WVD (PWVD)

\[ PW_s(t, \omega) = \frac{1}{2\pi} \int s(t + \tau/2)s^*(t - \tau/2)h(t)e^{-j\omega\tau}d\tau \]
Pseudo-WVD
Pseudo-WVD

• However, the consequence of this improved readability is that many properties of the WVD are lost:
  
  • The marginal properties
  • The unitarity
  • The frequency-support conservation

• The frequency-widths of the auto-terms are increased by this operation
Relationship between the WVD and the spectrogram

• The spectrogram can be expressed as a smoothing of the WVD

\[ P_{SP}(t, \omega) = \left| \frac{1}{\sqrt{2\pi}} \int s(\tau)h(\tau - t)e^{-j\omega \tau} d\tau \right|^2 = \iiint W_h(u - t, \theta - \omega)W_s(t, \omega)dud\theta \]

• The smoothing function \( \Phi(u, \theta) = W_h(u, \theta) \) is controlled only by the short-time window \( h(t) \)

• We can add another degree of freedom \( \Phi(t, \omega) = g(t)H(-\omega) \) allowing a progressive, independent control in both time and frequency of the smoothing applied to the WVD
Variations of WVD

WVD  PWVD  SPWVD
From the spectrogram to the WVD
The ambiguity function

• The **symmetrical ambiguity function (AF)**

\[ A_s(\theta, \tau) = \int s(u + \tau/2)s^*(u - \tau/2)e^{j\theta u} \, du \]

• The AF is a measure of the time-frequency correlation of the signal \( s \)

• The ambiguity function is the 2-D Fourier transform of the WVD. Consequently for the AF, a dual property corresponds to nearly all the properties of the WVD
Lecture 7 – Summary and Applications of JTFA

Properties of the AF

\[ AF(\theta, \tau) \]

\[ s^*(t - \frac{1}{2} \tau) s(t + \frac{1}{2} \tau) \]

\[ WVD(t, \omega) \]

Ambiguity function

Wigner-Ville distribution

The auto-terms are around the origin in the ambiguity function domain
Properties of the AF

- Wigner-Ville weighting function
- Wigner-Ville distribution
- Spectrogram weighting function
- Spectrogram
- SP-WV weighting function
- Smoothed-pseudo-WVD
Cohen’s class

• The approach characterizes time-frequency distributions by an auxiliary function called the **kernel function**

• The properties of a particular distribution are reflected by simple constraints on the kernel

• Therefore, it is possible to choose those kernels with prescribed, desirable properties

• This general class can be described in a number of different ways
General description of Cohen’s class

- **All** time-frequency representations can be obtained from

\[
C_s(t, \omega) = \frac{1}{4\pi^2} \iiint \phi(\theta, \tau) e^{j\theta(u-t)} s(u + \tau/2) s^*(u - \tau/2) e^{-j\omega \tau} d\theta d\tau d\omega
\]

or equivalently

\[
C_s(t, \omega) = \frac{1}{4\pi^2} \iiint \phi(\theta, \tau) e^{j\tau(u-\omega)} s^*(u + \theta/2) s(u - \theta/2) e^{-j\theta \tau} d\theta d\tau du
\]

where \( \phi(\theta, \tau) \) is the **kernel function**
General description of Cohen’s class

\[ s^*(t - \frac{1}{2} \tau)s(t + \frac{1}{2} \tau) \]

\[ F_t \]
\[ AF(\theta, \tau) \]
\[ \phi(\theta, \tau) \]
\[ \phi \cdot AF \]
\[ F_{\tau}F_{\theta}^{-1} \]
\[ TFR_{\phi} \]

\[ C(t, \omega, \phi) = \frac{1}{4\pi^2} \int \int \int \phi(\theta, \tau)s^*(u - \frac{1}{2} \tau)s(u + \frac{1}{2} \tau)e^{-j\theta t - j\omega \tau + j\phi u} d\tau d\theta \]
Other important energy distributions

• The Rihaczek distribution
• The Margenau-Hill distribution
• The Page distribution
Other important energy distributions

- **Joint-smoothings of the WVD**: the following distributions correspond to particular cases of the Cohen’s class for which the parameterization function depends only on the product of the variables $\tau$ and $\theta$

  $$\phi(\theta, \tau) = f(\theta \tau)$$

  where $f$ is a decreasing function such that $f(0) = 1$

- A direct consequence of this definition is that the marginal properties will be respected
Other important energy distributions

- Since $f$ is a decreasing function, $\phi$ is a low-pass function, and thus, this parameterization function will reduce the interferences.

- That is why these distributions are also known as the Reduced Interference Distributions (RID)

- Some examples are: Choi-Williams, Born-Jordan and Zhao-Atlas-Marks distributions
Other important energy distributions
Other important energy distributions
Other important energy distributions

<table>
<thead>
<tr>
<th>Reference</th>
<th>Kernel $\phi(\theta,\tau)$</th>
<th>Distribution $P(t,\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wigner [199], Ville [194]</td>
<td>$1$</td>
<td>$\frac{1}{2\pi} \int \frac{e^{-i\omega}}{1} s^*(t - \frac{1}{2}\tau) s(t + \frac{1}{2}\tau) d\tau$</td>
</tr>
<tr>
<td>Margenau and Hill [133]</td>
<td>$\cos \frac{1}{2} \theta \tau$</td>
<td>$\text{Re} \left( \frac{1}{\sqrt{2\pi}} \int s(t) e^{-i\omega} S^*(\omega) \right)$</td>
</tr>
<tr>
<td>Kirkwood [107], Rihaczek [167]</td>
<td>$e^{i\tau/2}$</td>
<td>$\frac{1}{\sqrt{2\pi}} \int s(t) e^{-i\omega} S^*(\omega)$</td>
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<tr>
<td>sinc [58]</td>
<td>$\frac{\sin \theta \tau}{\theta \tau}$</td>
<td>$\frac{1}{4\pi a} \int \left( \int_0^{\tau} \frac{1}{t} e^{-i\omega} \int_{t-a\tau}^{t+a\tau} s^*(u - \frac{1}{2}\tau) s(u + \frac{1}{2}\tau) du d\tau \right)^2$</td>
</tr>
<tr>
<td>Page [152]</td>
<td>$e^{i\theta/2 - i\tau / a}$</td>
<td>$\frac{\partial}{\partial t} \left( \frac{1}{\sqrt{2\pi}} \int \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \int_0^{\infty} s(t') e^{-i\omega t'} dt' \right)^2$</td>
</tr>
<tr>
<td>Choi and Williams [51]</td>
<td>$e^{-\theta^2/4}$</td>
<td>$\frac{1}{4\pi^{3/2}} \int \int \int \frac{1}{\sqrt{2\pi}} e^{-\omega^2 - x^2 / 4\tau^2 - i\tau \omega} s^*(u - \frac{1}{2}\tau)$</td>
</tr>
<tr>
<td>Spectrogram</td>
<td>$h^*(u - \frac{1}{2}\tau) e^{-i\omega u}$</td>
<td>$\left( \frac{1}{\sqrt{2\pi}} \int e^{-i\omega} s(\tau) h(\tau - t) d\tau \right)^2$</td>
</tr>
</tbody>
</table>
Summary

• The Cohen’s class gather all the quadratic time-frequency distributions covariant by shifts in time and in frequency

• It offers a wide set of powerful tools to analyze non-stationary signals. The basic idea is to devise a joint function of time and frequency that describes the energy density or intensity of a signal simultaneously in time and in frequency

• The most important element of this class is probably the Wigner-Ville distribution, which satisfies many desirable properties
Summary

• Since these distributions are quadratic, they introduce cross-terms in the time-frequency plane which can disturb the readability of the representation.

• One way to attenuate these interferences is to smooth the distribution in time and in frequency, according to their structure.

• The consequence of this is a decrease of the time and frequency resolutions, and more generally a loss of theoretical properties.
References and further reading


• The Time Frequency Toolbox tutorial (http://tftb.nongnu.org/tutorial.pdf)

• Slides from JTFA course by Dario Farina
Applications: Time Frequency Distribution of Cardiac Sounds
Applications: Spectrograms of EEG signals from the cortex of the rat

- Spontaneous activity
- Low intensity stimulation
Applications: Fatigue assessment using WVD of surface EMG
Applications: Choi-Williams distribution applied to a burst of surface EMG
Applications of JTFA to biomedical signal analysis

• Automatic detection of conduction block based on time-frequency analysis of unipolar electrograms

• Time-frequency analysis of movement-related spectral power in EEG during repetitive movements: a comparison of methods

• Adaptive time-frequency analysis of knee joint vibroarthrographic signals for noninvasive screening of articular cartilage pathology

• Instantaneous parameter estimation in cardiovascular time series by harmonic and time-frequency analysis