Notes about expansions in general and Fourier methods in particular

We are used to looking at signals in the time domain, because we feel that it is in time (or in space) where signals naturally occur. Lung sounds, picked up with a stethoscope, pass by in time and so do signals of nerves, the brain and the heart.

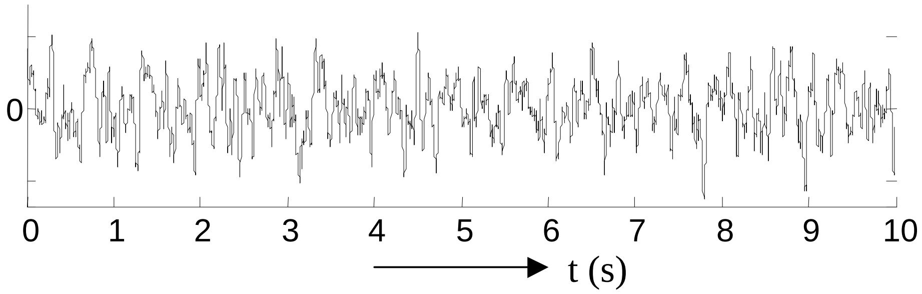

Figure 1 One channel of EEG in the time domain

But certain aspects or features of a signal can better be seen or analysed in other domains. The frequency domain, where the signals are seen in terms of trigonometric functions (e.g., sine waves), is very useful when signals are analyzed in relation to linear systems. The effect of linear filters is very easily described in the frequency domain, much easier than in the time domain, and, more importantly, very often filter requirements to be used in filter design are completely described in the frequency domain. A typical exception is the impulse response of the filter for which sometimes restrictions have to be defined, especially when there are impulse-like features in the signal, such as the QRS complex in an ECG. In the frequency domain it is sometimes also easier to perform certain signal processing methods or to describe certain features of the signal. For example, “more than 90% of the power of the signal is between 500 and 1000 Hz”, is a very simple statement about the signal in the frequency domain, but it would be very difficult to describe the same property of that signal in the time domain. On the other hand, it is very easy to describe something like “the H-reflex occured 70 ms after stimulation” in the time domain, but it would be more difficult to describe the same property in the frequency domain.

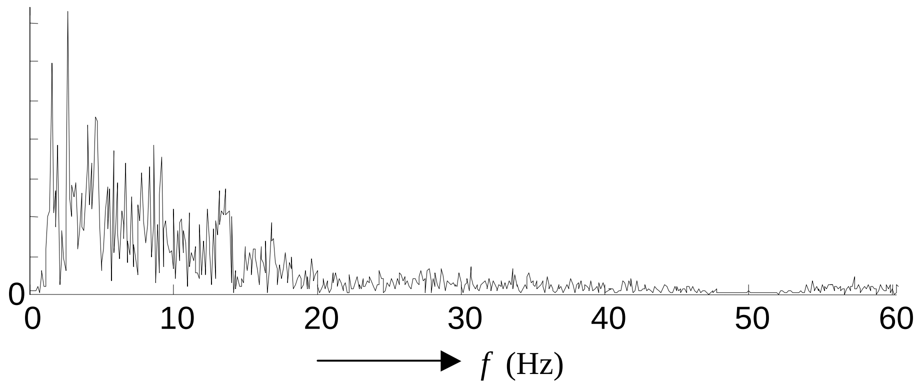

Figure 2 Magnitude spectrum of the signal from figure 1. Most of the signal is between 1 Hz and 20 Hz. Note the zero magnitude between 48 and 52 Hz: apparently a notch filter was used to attenuate the 50 Hz line interference (try to see that in fig. 1 !).

The transformation of a signal from the time domain to the frequency domain and back is, therefore, a very popular linear transformation. An example is the Fourier Transform.

Continuous expansion

Before embarking on Fourier transforms, let us first look at the more general idea: In general, a continuous-time signal $x(t)$ can be decomposed (expanded, transformed) linearly on a continuous set of basis functions $\phi_s(t)$, or written slightly differently, $\phi(s,t)$:

(1.) $x(t) = \int X(s)\,\phi(s,t)\,ds$ (reconstruction formula)

Basically, this means that the signal $x(t)$ is written as a combination of other functions $\phi_s(t)$ with various values of the parameter s and various amplitudes $X(s)$. In other words, the signal $x(t)$ can be constructed using basis functions $\phi(s,t)$.

An example is the <u>inverse</u> Fourier transform: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$, showing how the signal can be (re)constructed from the Fourier coefficients $X(j\omega)$ using harmonic basis functions $\phi(j\omega,t) \equiv e^{j\omega t}$ with various frequencies ω and amplitudes $X(j\omega)$.

Note that s does not have to be continuous, but can be either discrete (meaning that there is a discrete set of continuous basis functions, in which case the integral in equation 1 is changed into a summation) or continuous (with a continuous set of continuous basis functions). The (inverse) Fourier transform is an example of the continuous case, whereas the Fourier Series is an example of the discrete case:
$x(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X_n e^{j2\pi n/T}$ (where $x(t)$ is now periodic with period T).

Thus, $x(t)$ can be written as a linear combination of basis functions $\phi(s,t)$. We "only" have to find the coefficients $X(s)$.

It is important to recognize three points:

- The basis functions have to be able to produce the signal $x(t)$, and we simply assume that they are, otherwise we would not call them *basis* functions: we assume here that they form a complete basis for the space in which $x(t)$ exists and we assume that the basis functions are linearly independent: the basis is not redundant.
- For a given basis, $\{\phi\}$, (i.e., for a given set of basis functions $\phi(s,t)$) it is sufficient to know the $X(s)$ in order to know the whole signal $x(t)$: If we know both $X(s)$ and the basis, then we know how to construct $x(t)$ using the basis functions. In other words, $X(s)$ completely represents the signal $x(t)$, just in a different domain (s instead of t). And all the properties of $x(t)$ can, in principle, be derived from $X(s)$, and vice versa.
- For a given signal $x(t)$, and a given basis, $\{\phi\}$, it would be nice to have a method to calculate the corresponding $X(s)$. In other words: we need a way to transform $x(t)$ to $X(s)$.

The fact that $x(t)$ and $X(s)$ are two representations of the same thing suggests that there is a certain symmetry between them: we must be able not only to write $x(t)$ as a function of $X(s)$, but also vice versa:

(2.) $X(s) = \int x(t)\,\phi^*(s,t)\,dt$ (decomposition, analysis, expansion formula)

Examples are the Laplace transform: $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$, where $\phi^*(s,t) \equiv e^{-st}$, and the Fourier transform $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$, where $\phi^*(s,t) \equiv e^{-j\omega t}$.

In equation (2.) the unknowns are now the functions $\phi^*(s,t)$. We can obtain an important property of the functions $\phi(s,t)$ and $\phi^*(s,t)$ by substituting eq. (1.) in eq. (2.):

$$X(s) = \int \left(\int X(\sigma)\, \phi(\sigma,t)\, d\sigma \right) \phi^*(s,t)\, dt$$, which can be reorganized as

(3.) $$X(s) = \int X(\sigma) \left(\int \phi(\sigma,t) \phi^*(s,t)\, dt \right) d\sigma$$

Comparing this with the identity: $X(s) \equiv \int X(\sigma) \delta(s-\sigma)\, d\sigma$, which is the (implicit) definition of the Dirac function $\delta(s)$, we see that (3.) can be true only if

(4.) $$\int \phi(\sigma,t) \phi^*(s,t)\, dt = \delta(s-\sigma)$$

Equation (4.) means that the basis functions are so-called biorthogonal. If the set $\{\phi\}$ is orthonormal than it can be proved that $\phi^*(s,t) = \phi(s,t)$.

In the above exposure, the transform (2.) (including, e.g., the Fourier transform and the Laplace transform) is basically a change of basis, or a change of domain. Conceptually, it is a change of point of view: looking at the same thing from a different angle. And it should be very clear that, for example, the Fourier transform of a signal is still the same signal with exactly the same properties. It just looks different because you look at it in a different way, in a different basis, but all the things you want to do in the time domain can also be done in the frequency domain (for example: filtering). However, sometimes it is easier to do one thing in the time domain and easier to do something else in the frequency domain.

Besides the Laplace transform, the Fourier transform and Fourier series, also other standard transforms exist, most importantly, the Gabor transform and the wavelet transform (the latter actually being a whole class of various transforms). Historically, the Fourier series is the first example of a signal expansion. But that is not why it is such an important one: the harmonic sines and cosines are eigenfunctions of linear time-invariant systems. The only thing that may happen to the sine waves in a linear time-invariant system is that the amplitudes and the phases change. But otherwise, it is: "sine in, sine out".

It is important to understand a different interpretation of the formalism given above.
The general integral

(5.) $$< f, g > = \int f(t) g(t) dt$$

is the inner product of the functions *f* and *g* in the same way as the scalar product

(6.) $$\mathbf{v} \cdot \mathbf{w} = \sum v[n] w[n]$$

is the inner product of the vectors **v** and **w**.

Equations (1) and (2) thus describe the inner products of the signal with the respective basis functions. At the same time an inner product can be interpreted as the projection of one vector (or function) on the second vector (or function). This projection gives zero when the vectors (or functions) are orthogonal to each other, and a maximum result occurs when the vectors point exactly in the same direction (in parallel), or when the functions are identical.

Transforming a signal to a new basis is thus obtained by looking how similar the signal is to each of the basis functions of the new basis. The values for the similarity (the projection) of the signal to (on) each of the basis functions are then a new representation of the signal. Generally speaking we have to divide between the basis $\{\phi\}$ and its so-called dual basis $\{\phi^*\}$, but here we will work with orthogonal bases only.

> Taking the inner product of the signal with the respective basis functions is also exactly what is done in a Fourier transform: the signal is compared with a range of sines and cosines with different frequencies by taking the inner product of the signal with the complex function $e^{-j\omega t} \equiv \cos\omega t - j\sin\omega t$. In this way the signal is compared with cosines and sines of all radial frequencies ω. The resulting coefficients $X(j\omega)=\mathrm{Re}(X(j\omega))+j\mathrm{Im}(X(j\omega))$ show how well the signal fits with the cosines and sines for each of the values of ω.

Discrete expansion

Let us repeat the above arguments for the case where both *t* and *s* are discrete instead of continuous variables.
A discrete-time signal $x[n]$ ($n=0,\ldots,N-1$) can be expanded linearly on *N* basis functions $\phi_k[n]$, ($k=0,\ldots N-1$). In other words: we can build a signal $x[n]$ as a linear combination of a set of basis functions (reconstruction formula):

(7.) $$x[n] = \sum_{k=0}^{N-1} X[k]\phi_k[n]$$ (note the similarity with eq. (6.))

Thus, if we want to write a known signal $x[n]$ in terms of the basis functions $\phi_k[n]$ we have to find the coefficients $X[k]$.
Three points are important to recognize at this point:

- We assume that $\{\phi\}$ is a complete, linearly independent basis.
- For a given basis it is sufficient to know the $X[k]$ in order to be able to reconstruct the whole signal $x[n]$. In other words, $X[k]$ ($k=0,\ldots,N-1$) represents the signal $x[n]$ ($n=0,\ldots,N-1$).
- For a given signal $x[n]$, and a given basis $\{\phi\}$ it would be nice to have a method to calculate the corresponding $X[k]$. In other words: we need a way to transform $x[n]$ to $X[k]$.

The fact that $x[n]$ and $X[k]$ are two representations of the same signal suggests that there is a certain symmetry between them: we must be able not only to write $x[n]$ as a function of $X[k]$, but also vice versa:

(8.) $$X[k] = \sum_{n=0}^{N-1} x[n]\phi_k^*[n]$$

But now we have changed the problem from finding $X[k]$ to finding the (dual) basis $\{\phi_k^*\}$. We can try to find the solution to this problem by writing the $x[n]$ in equation (8). in terms of equation (7.):

$$X[k] = \sum_{n=0}^{N-1}\left(\sum_{m=0}^{N-1} X[m]\phi_m[n]\right)\phi_k^*[n],$$ which can be reorganized as

(9.) $$X[k] = \sum_{m=0}^{N-1} X[m]\sum_{n=0}^{N-1}\phi_m[n]\phi_k^*[n]$$

Equation (9.) can be true only if

(10.) $$\sum_{n=0}^{N-1}\phi_m[n]\phi_k^*[n] = \delta(m-k) = \begin{cases}1, & \text{for } m = k\\ 0, & \text{for } m \neq k\end{cases}$$

If n represents the discrete time variable and k the discrete frequency, then (7.) and (8.) form the Discrete Fourier Transform (DFT) pair if we take $\phi_k[n] \equiv e^{j2\pi kn/N}/N$, and $\phi_k^*[n] \equiv e^{-j2\pi kn/N}$. It can easily be shown that this pair of basis functions satisfy equation (10):

$$\sum_{n=0}^{N-1}\frac{1}{N}e^{j2\pi mn/N}e^{-j2\pi kn/N} = \frac{1}{N}\sum_{n=0}^{N-1}e^{j2\pi(m-k)n/N} = \begin{cases}1, & \text{for } m-k = pN,\ p \text{ integer}\\ 0, & \text{otherwise}\end{cases}$$

See exercise 1.

Usually the DFT is calculated using a fast algorithm, the Fast Fourier Transform (FFT), which is a standard tool (also in MatLab).

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An overview of the different Fourier methods:

	Continuous Frequency	*Discrete Frequency*
Continuous Time	Fourier transform (FT)	Fourier Series (FS) periodicity in time
Discrete Time	Discrete-time Fourier transform (DTFT) periodicity in frequency	**Discrete Fourier transform (DFT)** periodicity in time and frequency

Jean Baptiste Joseph Fourier (1768 – 1830)
Fourier studied heat conduction from a mathematical point of view. He derived the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions (Fourier Series) in 1807. However, his work was poorly accepted and even controversial at the time.

In Grenoble, where he was appointed by Napoleon as the Prefect of the Department of Isere, Fourier began his work on heat conduction around 1804. By 1807 he had completed his memoir *On the Propagation of Heat in Solid Bodies.* The memoir was read to the Paris Institute in December 1807. Nowadays this memoir is considered to be a very important contribution to mathematics as well as the theory of heat conduction, but at the time it caused controversy. One of the main objections, made by Lagrange and Laplace in 1808, was to Fourier's expansions of functions as trigonometrical series, (now called Fourier series).

For the 1811 mathematics prize, the Paris Institute had chosen the subject of the propagation of heat in solid bodies. Fourier submitted his 1807 memoir together with additional work. A committee consistingof Lagrange, Laplace, Malus, Haüy and Legendre, awarded Fourier the prize. But their report states:- ... *the manner in which the author arrives at these equations is not exempt of difficulties and his analysis to integrate them still leaves something to be desired* …. In other words: the committee still was not happy with the Fourier series.

During Fourier's eight last years in Paris he resumed his mathematical researches and published a number of papers. But his theory of heat continued to provoke controversy. Biot claimed priority over Fourier, a claim which Fourier had little difficulty showing to be false. Poisson, however, attacked both Fourier's mathematical techniques and also claimed to have an alternative theory.

Fourier's work is now the basis for a large number of methods in mathematics, science and engineering.