Opgave 1. (15%)

Input-output relation er for et LTI system er

$$y[n] = x[n] + 2x[n+1] - 3x[n-1]$$

1.a. Er systemet kausalt?

Nej, y er afhængigt af fremtidige værdier af x, altså x[n+1]

Systemets input er x[n] er en impuls respons

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

1.b. Beregn y[n] for n={-2 -1 0 1 2 }

$$y[-2] = 1 * 0 + 2 * 0 - 3 * 0 = 0$$

$$y[-1] = 1 * 0 + 2 * 1 - 3 * 0 = 2$$

$$y[0] = 1 * 1 + 2 * 0 - 3 * 0 = 1$$

$$y[1] = 1 * 0 + 2 * 0 - 3 * 1 = -3$$

$$y[2] = 1 * 0 + 2 * 0 - 3 * 0 = 0$$

1.c Er systemet stabilt?

Ja, impulsresponsen er absolut summerbar.

1.d Bestem H(z)

$$Y(z) = X(z) + 2z^{+1}X(z) - 3z^{-1}X(z)$$

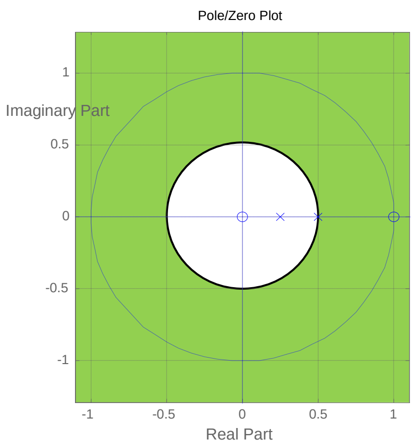
$$\boldsymbol{H(z) = \frac{Y(z)}{X(z)} = 2z^{+1} + 1 - 3z^{-1}}$$

Opgave 2. (35%)

Et kausalt systems overførselsfunktionen er givet ved

$$H(z) = \frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

2.a Skitser poler, nulpunkter og konvergensområdet



Da systemet er kausalt er ROC > den største pol.

2.b Er systemet stabilt?

Ja, enhedscirklen ligger i ROC.

2.c Find systemets impuls respons h[n]

Brug partialbrøksopspaltning

$$H(z) = \frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})} = \frac{A}{(1 - \frac{1}{2}z^{-1})} + \frac{B}{(1 - \frac{1}{4}z^{-1})}$$

Find A
Gang med $(1 - \frac{1}{2}z^{-1})$

$$\frac{1 - z^{-1}}{(1 - \frac{1}{4}z^{-1})} = A + \frac{B\,(1 - \frac{1}{2}z^{-1})}{(1 - \frac{1}{4}z^{-1})}$$

Sæt $z = \frac{1}{2}$

$$\frac{1 - 2}{(1 - \frac{1}{4}2)} = -2 = A$$

Find B
Gang med $(1 - \frac{1}{4}z^{-1})$

$$\frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})} = \frac{A\,(1 - \frac{1}{4}z^{-1})}{(1 - \frac{1}{2}z^{-1})} + B$$

Sæt $z = \frac{1}{4}$

$$\frac{1-4}{(1-\frac{1}{2}4)} = 3 = B$$

Dermed

$$H(z) = \frac{1-z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{-2}{(1-\frac{1}{2}z^{-1})} + \frac{3}{(1-\frac{1}{4}z^{-1})}$$

Ved tabel opslag får vi

$$\boldsymbol{h[n] = -2\left(\frac{1}{2}\right)^{n} u[n] + 3\left(\frac{1}{4}\right)^{n} u[n]}$$

2.d Find outputtet y[n] hvis inputtet x[n] er en step funktion:

$$x[n] = u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Vi ved at

$Y(z) = H(z)X(z)$ *og* $[n] \underset{z}{\rightarrow} \frac{1}{1-z^{-1}}$

$$Y(z) = \frac{1-z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}\left(\frac{1}{1-z^{-1}}\right)$$

Dermed får vi

$$Y(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

Brug partialbrøksopspaltning

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{A}{(1-\frac{1}{2}z^{-1})} + \frac{B}{(1-\frac{1}{4}z^{-1})}$$

Find A

Gang med $(1-\frac{1}{2}z^{-1})$

$$\frac{1}{(1-\frac{1}{4}z^{-1})} = A + \frac{B\,(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{4}z^{-1})}$$

Sæt $z = \frac{1}{2}$

$$\frac{1}{(1-\frac{1}{4}2)} = 2 = A$$

Find B

Gang med $(1-\frac{1}{4}z^{-1})$

$$\frac{1}{(1-\frac{1}{2}z^{-1})} = \frac{A\,(1-\frac{1}{4}z^{-1})}{(1-\frac{1}{2}z^{-1})} + B$$

Sæt $z = \frac{1}{4}$

$$\frac{1}{(1-\frac{1}{2}4)} = -1 = B$$

Dermed

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{2}{(1-\frac{1}{2}z^{-1})} + \frac{-1}{(1-\frac{1}{4}z^{-1})}$$

Ved tabel opslag får vi

$$\boldsymbol{h[n] = 2\left(\frac{1}{2}\right)^n u[n] - 1\left(\frac{1}{4}\right)^n u[n]}$$

Opgave 3. (10%)

En 10-bits analog til digital (AD) konverter giver ved 2.5 og 3.3 volt input henholdsvis 767 og 849 som output. AD konverteren tager input, der ligger mellem V_{R+} og V_{R-} (der begge er heltal).

3.a. Hvad er AD konverterens arbejdsområde? (vis beregningsmetode)

Sammenhængen mellem ADC output og konverterens arbejdsområde:

$$\mathrm{N_{ADC}} = (2^n - 1)\frac{V_{in}-V_{R-}}{V_{R+}-V_{R-}}$$ *,der kan også anvendes* $(2^n)\frac{V_{in}-V_{R-}}{V_{R+}-V_{R-}}$ *(det er producent afhængigt)*

Vi ved altså:

$$767 = 1023 * \left[\frac{2.5-V_{R-}}{V_{R+}-V_{R-}}\right] \qquad 849 = 1023 * \left[\frac{3.3-V_{R-}}{V_{R+}-V_{R-}}\right]$$

Vi skal isolere V_{R-} *og* V_{R+} *ud fra ovenstående (2 ligninger med 2 ubekendte):*

$$\frac{767}{1023} * (V_{R+} - V_{R-}) = (2.5 - V_{R-}) \qquad \frac{849}{1023} * (V_{R+} - V_{R-}) = (3.3 - V_{R-})$$

$$\frac{767}{1023} * V_{R+} - \frac{767}{1023} * V_{R-} = 2.5 - V_{R-} \qquad \frac{849}{1023} * V_{R+} - \frac{849}{1023} * V_{R-} = 3.3 - V_{R-}$$

$$\frac{767}{1023} * V_{R+} + \frac{256}{1023} * V_{R-} = 2.5 \qquad \frac{849}{1023} * V_{R+} + \frac{174}{1023} * V_{R-} = 3.3$$

$$\frac{767}{1023} * V_{R+} + \frac{256}{1023} * V_{R-} = 2.5 \qquad \frac{849}{1023} * V_{R+} + \frac{174}{1023} * V_{R-} = 3.3$$

Nu har vi to muligheder: tag lommeregneren eller løs opgaven i hånden:

$\frac{767}{1023} = \mathrm{A}$ $\quad \frac{256}{1023} = B$ $\quad \frac{849}{1023} = \mathrm{C}$ $\quad \frac{174}{1023} = \mathrm{D}$ $\quad$ 2.5=E $\quad$ 3.3=F $\quad V_{R+} = X$ $\quad V_{R-} = Y$

Eq1: $\mathrm{A}X + BY = E$ $\qquad$ Eq2: $\mathrm{C}X + DY = F$

Find udtryk for X i Eq1: $\mathrm{X} = \frac{E-BY}{A}$

Indsæt udtryk for X i Eq2: $Y = \frac{F}{D} - \frac{C}{D}X = \frac{F}{D} - \frac{C}{D} * \frac{E-BY}{A} = \frac{F}{D} - \frac{C}{DA} * (E - BY)$

$$Y = \frac{F}{D} - \frac{CE}{DA} + \frac{CB}{DA}Y$$

$$Y * \left(1 - \frac{CB}{DA}\right) = \frac{F}{D} - \frac{CE}{DA}$$

$$Y = \left(\frac{F}{D} - \frac{CE}{DA}\right) / \left(1 - \frac{CB}{DA}\right)$$

$$Y = -4.9829$$

$$\mathrm{X} = \frac{E - BY}{A} = 4.9976$$

AD konverterens arbejdsområde er altså:

$$V_{R+} = 5V \qquad V_{R-} = -5V$$

Der er afrundingsfejl på udregningen da 767 og 849 (output fra ADC) ikke er en 100% nøjagtig repræsentation af 2.5V og 3.3V (input til ADC). Dette skyldes kvantiseringsfejl.

3.b. Hvor stor er kvantiseringsfejlene på de to målinger?
(angivet i mV med 2 betydende cifre)

$$\mathrm{N_{ADC}} = \left[(2^n - 1)\frac{V_{in} - V_{R-}}{V_{R+} - V_{R-}}\right]$$

Find udtryk for V_{in}:
(Resultatet vil afvige, da N_{adc} skal være et heltal [ ]…disse parenteser betyder afrunding "eng: rounding".
Der kunne også have været brugt $\lfloor\ \rfloor$ … disse parenteser betyder nedrunding "eng: truncation".)

$$V_{in} + kvantiseringsfejl = \frac{\mathrm{N_{ADC}} * (V_{R+} - V_{R-})}{(2^n - 1)} + V_{R-}$$

$$kvantiseringsfejl = \frac{\mathrm{N_{ADC}} * (V_{R+} - V_{R-})}{(2^n - 1)} + V_{R-} - V_{in}$$

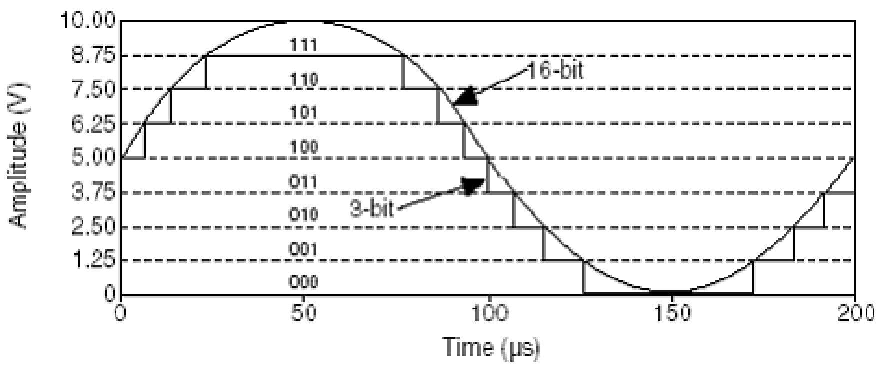
Kvantiseringsfejl for 2.5V (angivet i mV med 2 betydende cifre): $\frac{767*10}{1023} - 5 - 2.5 = -2.4mV$

Kvantiseringsfejl for 3.3V (angivet i mV med 2 betydende cifre): $\frac{849*10}{1023} - 5 - 3.3 = -0.88mV$
(0 tæller ikke med som betydende ciffer)

3.c. Hvad er den maksimale kvantiseringsfejl på en n-bits AD konverter?
(angiv symbolsk svar)

Q= arbejdsområde/2^N ,hvor Q er opløsning i volt per ADC trin og N er antallet af bit

Q bliver også omtalt som LSB (Least Significant Bit) og dét er den maksimale kvantiseringsfejl på en N-bits AD konverter. Den maksimale kvantiseringsfejl bliver angivet som 1 lsb (ved truncation) eller +- ½ lsb (ved rounding).



I ovenstående eksempel er LSB = 1.25V (når N=3 og arbejdsområde=10V). Hele området fra 0V til 1.25V vil give et ADC output på 0. Der er alstå her tale om "truncation" og den maksimale kvantiseringsfejl fås lige inden signalet når 1.25V (altså 1 LSB).

Opgave 4. (20%)

Complex functions:
Given the following function:

$$f(z) = \frac{\cosh 8z}{z}$$

4.a. Expand the function in a Laurent series that converges in a ring and determine the precise region of convergence.

4.b. Determine the singularities of f(z) and classify them.

4.c. Determine the residue of f(z) at z=0.

4.d. Evaluate the real integral $\int_{-\infty}^{\infty} \frac{\cosh 8x}{x} dx$

Opgave 5. (20%)

Linear Algebra:
Given the following matrix:

$$A = \begin{bmatrix} -6 & -6 & 10 \\ -5 & -5 & 5 \\ -9 & -9 & 13 \end{bmatrix}$$

(Show all relevant intermediate results – that means **no** calculator-only type of answers)

5.a. Find the characteristic polynomial of **A**.

5.b. Find the characteristic equation of **A**.

5.c. Find the eigenvalues of **A**. What is their multiplicity?

5.d. Find the corresponding eigenvectors and determine their multiplicity.

5.e. Diagonalize **A** using an eigenbasis of **A**.

PRØVEEKSAMEN Nr. 2

Opgave 4:

4.a. We use the series $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$

$$f(z) = \frac{\cosh 8z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(8z)^{2n}}{(2n)!} = \frac{1}{z}\left(1 + \frac{64z^2}{2!} + \frac{4096 z^4}{4!} + \cdots\right)$$

$$\frac{\cosh 8z}{z} = \frac{1}{z} + \frac{64z}{2!} + \frac{4096 z^3}{4!} + \cdots$$

$0 < |z| < R = \infty$ (Region of convergence)

4.b. Looking at the negative powers of z in the Laurent series, we can see that the function has a singularity at $z_0 = 0$

Since there is a finite number of terms (one) in the principal part of the series $\Rightarrow$

$\Rightarrow$ $z_0 = 0$ is a pole, of first order

4.c. $\operatorname{Res}_{z_0=0} f(z)$?

$\operatorname{Res}_{z=z_0} f(z) = b_1$ where b_1 is the coefficient of the first negative power of z in the Laurent series of $f(z)$. From 4.a.:

$$\operatorname{Res}_{z_0=0} f(z) = 1$$

4.d. The complex function $f(z) = \frac{\cosh 8z}{z}$ has a single pole on the real axis $\Rightarrow$ we must use:

$$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

Residue for all poles on upper-half plane

Residue for all poles on real axis

Therefore:

$$\int_{-\infty}^{\infty} \frac{\cosh 8x}{x} = \pi i \operatorname*{Res}_{z_0=0} f(z) = \pi i * 1$$ (There are no other poles)

$$\boxed{\int_{-\infty}^{\infty} \frac{\cosh 8x}{x} = \pi i}$$

Opgave 5:

5.a. $A = \begin{bmatrix} 5 & 4 \\ 4 & 11 \end{bmatrix}$

To find the characteristic polynomial we must find the characteristic determinant

$$\det(A - \lambda I) = \begin{bmatrix} 5-\lambda & 4 \\ 4 & 11-\lambda \end{bmatrix} = (5-\lambda)(11-\lambda) - 16 =$$

$$= 55 - 5\lambda - 11\lambda + \lambda^2 - 16 = \lambda^2 - 16\lambda + 39$$

$$\boxed{D(\lambda) = \lambda^2 - 16\lambda + 39}$$

5.b. The characteristic equation of A is $D(\lambda) = 0$ $\Rightarrow$

$$\boxed{\lambda^2 - 16\lambda + 39 = 0}$$

5.c. We have to solve the characteristic equation:

$$\lambda_{1,2} = \frac{16 \pm \sqrt{256 - 156}}{2} = \frac{16 \pm 10}{2} \quad \begin{array}{l} \nearrow \boxed{\lambda_1 = 13} \\ \searrow \boxed{\lambda_2 = 3} \end{array}$$

Both eigenvalues have an algebraic multiplicity $M = 1$

5.d. To find the eigenvectors we must solve:

$(A - \lambda I)\bar{X} = 0$

→ for $\lambda_1 = 13$

$$\begin{bmatrix} 5-13 & 4 \\ 4 & 11-13 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} -8X_1 + 4X_2 &= 0 \quad ① \\ 4X_1 - 2X_2 &= 0 \quad ② \end{aligned}\right\} \text{From } ② : X_2 = 2X_1 \; ③$$

Replacing ③ into ① $-8X_1 + 8X_1 = 0 \Rightarrow X_1$ can take any value ⇒

$\left.\begin{aligned} X_1 &= 1 \\ X_2 &= 2 \end{aligned}\right\}$ is a solution $\boxed{X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$ Geometric multiplicity = 1

→ for $\lambda_2 = 3$

$$\begin{bmatrix} 5-3 & 4 \\ 4 & 11-3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} 2X_1 + 4X_2 &= 0 \quad ① \\ 4X_1 + 8X_2 &= 0 \quad ② \end{aligned}\right\} \text{From } ① : X_1 = -2X_2 \; ③$$

Replacing ③ into ② $-8X_1 + 8X_2 = 0 \Rightarrow X_2$ can take any value ⇒

$\left.\begin{aligned} X_1 &= -2 \\ X_2 &= 1 \end{aligned}\right\}$ is a solution $\boxed{X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$ Geometric multiplicity = 1

5.6. We must calculate:

$D = X^{-1} A X$ with X being the matrix that contains the eigenvectors of A as columns, i.e. an eigenbasis of A:

$$X = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$ We have to find X^{-1}

Using the Gauss-Jordan elimination method:

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array}\right] \quad \text{Row 2} - 2\,\text{Row 1}$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right] \quad \text{Row 2} / 5$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/5 & 2/5 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right] \quad \text{Row 1} + 2\,\text{Row 2} \qquad \Rightarrow X^{-1} = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

We multiply first:

$$X^{-1}A = \begin{bmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.4 \\ -0.4 & 0.2 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 2.6 & 5.2 \\ -1.2 & 0.6 \end{bmatrix}$$

And now the result with X:

$$X^{-1}AX = \begin{bmatrix} 2.6 & 5.2 \\ -1.2 & 0.6 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 0 & 3 \end{bmatrix}$$

which gives a diagonal matrix containing the eigenvalues of A in the diagonal

$$\boxed{D = \begin{bmatrix} 13 & 0 \\ 0 & 3 \end{bmatrix}}$$