2.1. (a) $T(x[n]) = g[n]x[n]$

- Stable: Let $|x[n]| \le M$ then $|T[x[n]| \le |g[n]|M$. So, it is stable if $|g[n]|$ is bounded.
- Causal: $y_1[n] = g[n]x_1[n]$ and $y_2[n] = g[n]x_2[n]$, so if $x_1[n] = x_2[n]$ for all $n < n_0$, then $y_1[n] = y_2[n]$ for all $n < n_0$, and the system is causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= g[n](ax_1[n] + bx_2[n] \\ &= ag[n]x_1[n] + bg[n]x_2[n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  So this is linear.
- Not time-invariant:

$$\begin{aligned} T(x[n-n_0]) &= g[n]x[n-n_0] \\ &\neq y[n-n_0] = g[n-n_0]x[n-n_0] \end{aligned}$$

  which is not TI.
- Memoryless: $y[n] = T(x[n])$ depends only on the n^{th} value of x, so it is memoryless.

(b) $T(x[n]) = \sum_{k=n_0}^{n} x[k]$

- Not Stable: $|x[n]| \le M \to |T(x[n])| \le \sum_{k=n_0}^{n} |x[k]| \le |n - n_0|M$. As $n \to \infty$, $T \to \infty$, so not stable.
- Not Causal: $T(x[n])$ depends on the future values of $x[n]$ when $n < n_0$, so this is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= \sum_{k=n_0}^{n} ax_1[k] + bx_2[k] \\ &= a\sum_{k=n_0}^{n} x_1[n] + b\sum_{k=n_0}^{n} x_2[n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  The system is linear.
- Not TI:

$$\begin{aligned} T(x[n-n_0]) &= \sum_{k=n_0}^{n} x[k-n_0] \\ &= \sum_{k=0}^{n-n_0} x[k] \\ &\neq y[n-n_0] = \sum_{k=n_0}^{n-n_0} x[k] \end{aligned}$$

  The system is not TI.
- Not Memoryless: Values of $y[n]$ depend on past values for $n > n_0$, so this is not memoryless.

(c) $T(x[n]) \sum_{k=n-n_0}^{n+n_0} x[k]$

- Stable: $|T(x[n])| \le \sum_{k=n-n_0}^{n+n_0} |x[k]| \le \sum_{k=n-n_0}^{n+n_0} x[k]M \le |2n_0 + 1|M$ for $|x[n]| \le M$, so it is stable.
- Not Causal: $T(x[n])$ depends on future values of $x[n]$, so it is not causal.

- Linear:

$$\begin{aligned} T(ax_1[n]+bx_2[n]) &= \sum_{k=n-n_0}^{n+n_0} ax_1[k]+bx_2[k] \\ &= a\sum_{k=n-n_0}^{n+n_0} x_1[k]+b\sum_{k=n-n_0}^{n+n_0} x_2[k] = aT(x_1[n])+bT(x_2[n]) \end{aligned}$$

  This is linear.
- TI:

$$\begin{aligned} T(x[n-n_0] &= \sum_{k=n-n_0}^{n+n_0} x[k-n_0] \\ &= \sum_{k=n-n_0}^{n} x[k] \\ &= y[n-n_0] \end{aligned}$$

  This is TI.
- Not memoryless: The values of $y[n]$ depend on $2n_0$ other values of x, not memoryless.

(d) $T(x[n]) = x[n-n_0]$

- Stable: $|T(x[n])| = |x[n-n_0]| \le M$ if $|x[n] \le M$, so stable.
- Causality: If $n_0 \ge 0$, this is causal, otherwise it is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n]+bx_2[n]) &= ax_1[n-n_0]+bx_2[n-n_0] \\ &= aT(x_1[n])+bT(x_2[n]) \end{aligned}$$

  This is linear.
- TI: $T(x[n-n_d] = x[n-n_0-n_d] = y[n-n_d]$. This is TI.
- Not memoryless: Unless $n_0 = 0$, this is not memoryless.

(e) $T(x[n]) = e^{x[n]}$

- Stable: $|x[n]| \le M$, $|T(x[n])| = |e^{x[n]}| \le e^{|x[n]|} \le e^M$, this is stable.
- Causal: It doesn't use future values of $x[n]$, so it causal.
- Not linear:

$$\begin{aligned} T(ax_1[n]+bx_2[n]) &= e^{ax_1[n]+bx_2[n]} \\ &= e^{ax_1[n]}e^{bx_2[n]} \\ &\ne aT(x_1[n])+bT(x_2[n]) \end{aligned}$$

  This is not linear.
- TI: $T(x[n-n_0]) = e^{x[n-n_0]} = y[n-n_0]$, so this is TI.
- Memoryless: $y[n]$ depends on the n^{th} value of x only, so it is memoryless.

(f) $T(x[n]) = ax[n]+b$

- Stable: $|T(x[n])| = |ax[n]+b| \le a|M|+|b|$, which is stable for finite a and b.
- Causal: This doesn't use future values of $x[n]$, so it is causal.
- Not linear:

$$\begin{aligned} T(cx_1[n]+dx_2[n]) &= acx_1[n]+adx_2[n]+b \\ &\ne cT(x_1[n])+dT(x_2[n]) \end{aligned}$$

  This is not linear.

- TI: $T(x[n-n_0]) = ax[n-n_0] + b = y[n-n_0]$. It is TI.
- Memoryless: $y[n]$ depends on the n^{th} value of $x[n]$ only, so it is memoryless.

(g) $T(x[n]) = x[-n]$

- Stable: $|T(x[n])| \le |x[-n]| \le M$, so it is stable.
- Not causal: For $n < 0$, it depends on the future value of $x[n]$, so it is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= ax_1[-n] + bx_2x[-n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  This is linear.
- Not TI:

$$\begin{aligned} T(x[n-n_0]) &= x[-n-n_0] \\ &\ne y[n-n_0] = x[-n+n_0] \end{aligned}$$

  This is not TI.
- Not memoryless: For $n \ne 0$, it depends on a value of x other than the n^{th} value, so it is not memoryless.

(h) $T(x[n]) = x[n] + u[n+1]$

- Stable: $|T(x[n])| \le M + 3$ for $n \ge -1$ and $|T(x[n])| \le M$ for $n < -1$, so it is stable.
- Causal: Since it doesn't use future values of $x[n]$, it is causal.
- Not linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= ax_1[n] + bx_2[n] + 3u[n+1] \\ &\ne aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  This is not linear.
- Not TI:

$$\begin{aligned} T(x[n-n_0] &= x[n-n_0] + 3u[n+1] \\ &= y[n-n_0] \\ &= x[n-n_0] + 3u[n-n_0+1] \end{aligned}$$

  This is not TI.
- Memoryless: $y[n]$ depends on the n^{th} value of x only, so this is memoryless.

2.2. For an LTI system, the output is obtained from the convolution of the input with the impulse response of the system:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

(a) Since $h[k] \ne 0$, for $(N_0 \le n \le N_1)$,

$$y[n] = \sum_{k=N_0}^{N_1} h[k]x[n-k]$$

The input, $x[n] \ne 0$, for $(N_2 \le n \le N_3)$, so

$$x[n-k] \ne 0, \text{ for } N_2 \le (n-k) \le N_3$$

Note that the minimum value of $(n - k)$ is N_2. Thus, the lower bound on n, which occurs for $k = N_0$ is

$$N_4 = N_0 + N_2.$$

Using a similar argument,

$$N_5 = N_1 + N_3.$$

Therefore, the output is nonzero for

$$(N_0 + N_2) \leq n \leq (N_1 + N_3).$$

(b) If $x[n] \neq 0$, for some $n_o \leq n \leq (n_o + N - 1)$, and $h[n] \neq 0$, for some $n_1 \leq n \leq (n_1 + M - 1)$, the results of part (a) imply that the output is nonzero for:

$$(n_o + n_1) \leq n \leq (n_o + n_1 + M + N - 2)$$

So the output sequence is $M + N - 1$ samples long. This is an important quality of the convolution for finite length sequences as we shall see in Chapter 8.

2.3. We desire the step response to a system whose impulse response is

$$h[n] = a^{-n}u[-n], \text{ for } 0 < a < 1.$$

The convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

The step response results when the input is the unit step:

$$x[n] = u[n] = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

Substitution into the convolution sum yields

$$y[n] = \sum_{k=-\infty}^{\infty} a^{-k}u[-k]u[n-k]$$

For $n \leq 0$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n} a^{-k} \\ &= \sum_{k=-n}^{\infty} a^{k} \\ &= \frac{a^{-n}}{1-a} \end{aligned}$$

For $n \geq 0$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{0} a^{-k} \\ &= \sum_{k=0}^{\infty} a^{k} \\ &= \frac{1}{1-a} \end{aligned}$$

(c) Let $x[n] = u[n]$ (unit step), then

$$X(z) = \frac{1}{1 - z^{-1}}$$

and

$$\begin{aligned} Y(z) &= X(z) \cdot H(z) \\ &= \frac{2z^{-1}}{(1 - z^{-1})(1 - 2z^{-1})(1 - 3z^{-1})}. \end{aligned}$$

Partial fraction expansion yields

$$Y(z) = \frac{1}{1 - z^{-1}} - \frac{4}{1 - 2z^{-1}} + \frac{3}{1 - 3z^{-1}}.$$

The inverse transform yields:

$$y[n] = u[n] - 4(2)^n u[n] + 3(3)^n u[n].$$

2.6. (a) The difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

Taking the Fourier transform of both sides,

$$Y(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega}] = X(e^{j\omega})[1 + 2e^{-j\omega} + e^{-j2\omega}].$$

Hence, the frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \end{aligned}$$

(b) A system with frequency response:

$$\begin{aligned} H(e^{j\omega}) &= \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}} \\ &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \end{aligned}$$

cross multiplying,

$$Y(e^{j\omega})[1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}] = X(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}],$$

and the inverse transform gives

$$y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3].$$

2.7. $x[n]$ is periodic with period N if $x[n] = x[n+N]$ for some integer N.

(a) $x[n]$ is periodic with period 12:

$$e^{j(\frac{\pi}{6}n)} = e^{j(\frac{\pi}{6})(n+N)} = e^{j(\frac{\pi}{6}n + 2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{\pi}{6}N, \text{for integers } k, N$$

Making $k = 1$ and $N = 12$ shows that $x[n]$ has period 12.

(b) $x[n]$ is periodic with period 8:

$$e^{j(\frac{3\pi}{4}n)} = e^{j(\frac{3\pi}{4})(n+N)} = e^{j(\frac{3\pi}{4}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{3\pi}{4}N, \text{for integers } k, N$$

$$\Longrightarrow N = \frac{8}{3}k, \text{for integers } k, N$$

The smallest k for which both k and N are integers are is 3, resulting in the period N being 8.

(c) $x[n] = [\sin(\pi n/5)]/(\pi n)$ is not periodic because the denominator term is linear in n.

(d) We will show that $x[n]$ is not periodic. Suppose that $x[n]$ is periodic for some period N:

$$e^{j(\frac{\pi}{\sqrt{2}}n)} = e^{j(\frac{\pi}{\sqrt{2}})(n+N)} = e^{j(\frac{\pi}{\sqrt{2}}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{\pi}{\sqrt{2}}N, \text{for integers } k, N$$

$$\Longrightarrow N = 2\sqrt{2}k, \text{for some integers } k, N$$

There is no integer k for which N is an integer. Hence $x[n]$ is not periodic.

2.8. We take the Fourier transform of both $h[n]$ and $x[n]$, and then use the fact that convolution in the time domain is the same as multiplication in the frequency domain.

$$\begin{aligned} H(e^{j\omega}) &= \frac{5}{1+\frac{1}{2}e^{-j\omega}} \\ Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\ &= \frac{5}{1+\frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1-\frac{1}{3}e^{-j\omega}} \\ &= \frac{3}{1+\frac{1}{2}e^{-j\omega}} + \frac{2}{1-\frac{1}{3}e^{-j\omega}} \\ y[n] &= 2(\frac{1}{3})^n u[n] + 3(-\frac{1}{2})^n u[n] \end{aligned}$$

2.9. (a) First the frequency response:

$$Y(e^{j\omega}) - \frac{5}{6}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{6}e^{-2j\omega}Y(e^{j\omega}) = \frac{1}{3}e^{-2j\omega}X(e^{j\omega})$$

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{\frac{1}{3}e^{-2j\omega}}{1-\frac{5}{6}e^{-j\omega}+\frac{1}{6}e^{-2j\omega}} \end{aligned}$$

Now we take the inverse Fourier transform to find the impulse response:

$$\begin{aligned} H(e^{j\omega}) &= \frac{-2}{1-\frac{1}{3}e^{-j\omega}} + \frac{2}{1-\frac{1}{2}e^{-j\omega}} \\ h[n] &= -2(\frac{1}{3})^n u[n] + 2(\frac{1}{2})^n u[n] \end{aligned}$$

2.13. Eigenfunctions of LTI systems are of the form α^n, so functions (a), (b), and (e) are eigenfunctions. Notice that part (d), $\cos(\omega_0 n) = .5(e^{j\omega_0 n} + e^{-j\omega_0 n})$ is a sum of two α^n functions, and is therefore not an eigenfunction itself.

2.14. (a) The information given shows that the system satisfies the eigenfunction property of exponential sequences for LTI systems for one particular eigenfunction input. However, we do not know the system response for any other eigenfunction. Hence, we can say that the system may be LTI, but we cannot uniquely determine it. $\Longrightarrow$ (iv).

(b) If the system were LTI, the output should be in the form of $A(1/2)^n$, since $(1/2)^n$ would have been an eigenfunction of the system. Since this is not true, the system cannot be LTI. $\Longrightarrow$ (i).

(c) Given the information, the system *may* be LTI, but does not have to be. For example, for any input other than the given one, the system may output 0, making this system non-LTI. $\Longrightarrow$ (iii). If it were LTI, its system function can be found by using the DTFT:

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1}{1 - \frac{1}{2}e^{-j\omega}} \\ h[n] &= (\frac{1}{2})^n u[n] \end{aligned}$$

2.15. (a) No. Consider the following input/outputs:

$$x_1[n] = \delta[n] \quad \Longrightarrow \quad y_1[n] = (\frac{1}{4})^n u[n]$$

$$x_2[n] = \delta[n-1] \quad \Longrightarrow \quad y_2[n] = (\frac{1}{4})^{n-1} u[n]$$

Even though $x_2[n] = x_1[n-1]$, $y_2[n] \neq y_1[n-1] = (\frac{1}{4})^{n-1}u[n-1]$

(b) No. Consider the input/output pair $x_2[n]$ and $y_2[n]$ above. $x_2[n] = 0$ for $n < 1$, but $y_2[0] \neq 0$.

(c) Yes. Since $h[n]$ is stable and multiplication with $u[n]$ will not cause any sequences to become unbounded, the entire system is stable.

2.16. (a) The homogeneous solution $y_h[n]$ solves the difference equation when $x[n] = 0$. It is in the form $y_h[n] = \sum A(c)^n$, where the c's solve the quadratic equation

$$c^2 - \frac{1}{4}c + \frac{1}{8} = 0$$

So for $c = 1/2$ and $c = -1/4$, the general form for the homogeneous solution is:

$$y_h[n] = A_1(\frac{1}{2})^n + A_2(-\frac{1}{4})^n$$

(b) Taking the z-transform of both sides, we find that

$$Y(z)(1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}) = 3X(z)$$

and therefore

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{3}{1 - 1/4z^{-1} - 1/8z^{-2}} \\ &= \frac{3}{(1 + 1/4z^{-1})(1 - 1/2z^{-1})} \\ &= \frac{1}{1 + 1/4z^{-1}} + \frac{2}{1 - 1/2z^{-1}} \end{aligned}$$